Linear Bounds on Circuit Complexity and Feebly One-way Permutations

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Plan

1. Introduction
2. Upper bounds on circuit complexity
3. Lower bounds on circuit complexity
4. Feebly one-way families of permutations

Motivation

Practical:

- Logical design synthesis: smaller circuits — better designs.

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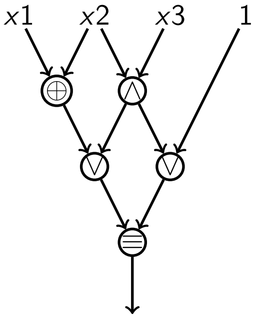
Theoretical:

- Circuits — very simple and natural model of computation. Many efforts spent — not too much known.

Boolean Circuits

- inputs: propositional variables $x_1, x_2, \ldots, x_n$ and constants 0, 1
- gates: binary functions
- fan-out of a gate is unbounded



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A boolean function is symmetric if its value depends on the sum of the input values only.

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Modular functions

Let $\mathrm{MOD}^n_{m,r}(x_1, \ldots, x_n) = 1 \iff \sum_{i=1}^n x_i \equiv r \pmod{m}$.

Example: $\mathrm{MOD}^n_{4,0}(x_1, \ldots, x_n = 1) \iff \sum_{i=1}^n x_i \equiv \{0, 4, 8, \ldots\}$

Stockmeyer's bounds for $\mathrm{MOD}_{4,0}^n$

- Stockmeyer constructed a circuit for $\mathrm{MOD}_{4,0}^n$ of size $2.5n + c$, using blocks with 6 inputs and 10 gates to add 4 new values to the remainder encoded by 2 bits and transfer the remainder encoded in 2 bits to the next block.

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- This matches the corresponding lower bound $2.5n + c$ proved by him.

Applying practice to theory

"For many important functions there is a large gap between known lower and upper bounds. It might be helpful to know optimal circuits for such functions at least for small values of input size. Knowing this could help us to understand the structure of optimal circuits for general functions."

Williams, R. (2008)

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By finding efficient small circuits we can obtain upper bounds on circiut complexity.

Main idea

Bruteforce search

- The number $F(n, t)$ of circuits of size $\leq t$ with n input variables does not exceed
$$\left(16(t+n+2)^2\right)^t .$$
Each of t gates is assigned one of 16 possible binary Boolean functions that acts on two previous nodes, and each previous node can be either a previous gate ($\leq t$ choices) or a variable or a constant ($\leq n+2$ choices).

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- To find Stockmeyer's block (6 inputs, 10 gates) a naive bruteforce over $\sim 1.4 * 10^{37}$ circuits will be needed.

Main idea

Given a function $f : \{0,1\}^n \to \{0,1\}^m$ (n, m are constants) we transform the fact "there exists a circuit of size m computing function f" into a CNF formula and use SAT-solvers to check its satisfiability.

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Enconding

- All possible underlying graphs of circuit
- All possibilities for functions computed by gates
- Which gates are outputs
- The particular function computed by circuit

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- New upper bound for $\mathrm{MOD}^n_{3,*}$: $5.5n + c$ in basis $U_2 = B_2 \setminus \{\oplus, \equiv\}$ (previous $7n + c$), using a block with 4 inputs and 11 gates.
- It is possible to prove exact bounds for circuits with ≤ 8 gates.

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- For $t = 2^n/(10n)$, $F(n,t)$ is approximately $2^{2^n/5}$, which is $\ll 2^{2^n}$.
- Thus, the circuit complexity of almost all Boolean functions on n variables is exponential in n. Still, we do not know any explicit function with super-linear circuit complexity.

Known Lower Bounds

	circuit size	formula size
full binary basis B_2	$3n - o(n)$ [Blum]	$n^{2-o(1)}$ [Nechiporuk]
basis $U_2 = B_2 \setminus \{\oplus, \equiv\}$	$5n - o(n)$ [Iwama et al.]	$n^{3-o(1)}$ [Hastad]
monotone basis $M_2 = \{\vee, \wedge\}$	exponential [Razborov; Alon, Boppana; Andreev; Karchmer, Wigderson]	

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- To avoid tricks like this one, we say that a function f is explicitly defined if $f^{-1}(1)$ is in NP.
- Usually, under a Boolean function f we actually understand an infinite sequence $\{f_n \mid n = 1, 2, \ldots\}$.

Known Lower Bounds for Circuits over B_2

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$2n - c$	[Schnorr, 74]
$2.5n - o(n)$	[Paul, 77]
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Gate Elimination

All the proofs are based on the so-called gate elimination method. This is essentially the only known method for proving lower bounds on circuit complexity.

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Remark

This method is very unlikely to produce nonlinear lower bounds.

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- Let $\mathrm{MOD}_{m,r}^n(x_1, \dots, x_n) = 1 \iff \sum_{i=1}^n x_i \equiv r \pmod{m}$.
- Then $\mathrm{MOD}_{3,r}^n, \mathrm{MOD}_{4,r}^n \in Q_{2,3}^n$, but $\mathrm{MOD}_{2,r}^n \notin Q_{2,3}^n$.

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Theorem

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- Thus, either x_i or x_j fans out to another gate P.
- By assigning this variable, we eliminate at least two gates and get a subfunction from $Q^{n-1}_{2,3}$. □

AND-type Gates vs XOR-type Gates

Binary functions

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Remark

Optimal circuits contain AND- and XOR-type gates only, as constant and degenerate gates can be easily eliminated.

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- That is why, in particular, the current record bounds for circuits over $U_2 = B_2 \setminus \{\oplus, \equiv\}$ are stronger than the bounds over B_2.
- Usually, the main bottleneck of a proof based on gate elimination is a circuit whose top contains many XOR-type gates.

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Lemma (Degree lower bound)

Any circuit computing f contains at least $\deg(\tau(f)) - 1$ AND-type gates.

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Definition

For a circuit C, let $A(C)$ and $X(C)$ denote the number of AND- and XOR-type gates in C, respectively. Let also $\mu(C) = 3X(C) + 2A(C)$.

An Improved Lower Bound

Lemma

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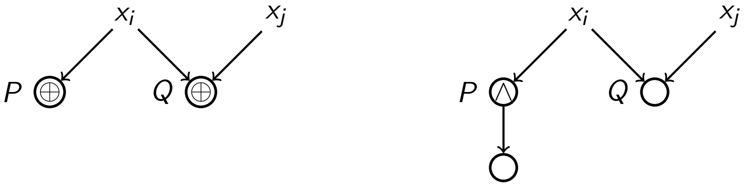
- There are two cases:



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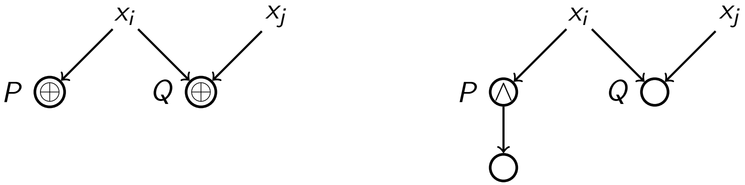

- In both cases, we can assign x_i a constant such that μ is reduced at least by 6. □

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Let C be an optimal circuit computing f.

$7n/3$ Lower Bound

Lemma

Let $f \in Q^n_{2,3}$ and $\deg(\tau(f)) \geq n - c$, then $C(f) \geq 7n/3 - c'$.

Proof

Let C be an optimal circuit computing f.

$$\frac{\begin{aligned} 3X(C) + 2A(C) &\geq 6n - 24 \\ A(C) &\geq n - c - 1 \end{aligned}}{3C(f) = 3X(C) + 3A(C) \geq 7n - 25 - c}$$

□

One-way permutations w.r.t circuit complexity

S_{2^n} is the subset of $B_{n,n}$ (the set of all boolean functions $f : \{0,1\}^n \to \{0,1\}^n$) containing all $2^n!$ invertible functions.
Any sequence $f_1, f_2, \ldots$ of functions $f_i \in S_{2^i}$ — a family of permutations denoted by $\{f_n\}$.

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$$M_F(f_n) = C(f_n^{-1})/C(f_n)$$

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This can be compared with the measure of practical one-wayness:

$$M_P(f_n) = \log_2[C(f_n^{-1})]/\log_2[C(f_n)]$$

One-way permutations w.r.t circuit complexity

A family of permutations $\{f_n\}$ is said to be feebly-one-way of order k, for some constant $k > 1$, if

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These definitions imply $C(f_n^{-1}) \sim k \cdot C(f_n)$ and $C(f_n^{-1}) = [C(f_n)]^{k \pm o(1)}$ respectively.

A linear family with feeble one-wayness of order $\frac{3}{2}$

Let's define ϕ_n, for $n \geq 3$ as a linear function:

$$\phi_n([x_1, \ldots, x_n]) = [y_1, \ldots, y_n]$$

where

$$y_i(x) = x_i \oplus x_{i+1} \qquad \text{for i} \neq n$$

$$y_n(x) = x_1 \oplus x_{\lceil n/2 \rceil} \oplus x_n$$

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The inverse function ϕ_n^{-1} is given by:

$$x_i(y) = (y_1 \oplus \cdots \oplus y_{i-1}) \oplus (y_{\lceil n/2 \rceil} \oplus \cdots \oplus y_n) \qquad i \leq \lceil n/2 \rceil$$

$$x_i(y) = (y_1 \oplus \cdots \oplus y_{\lceil n/2 \rceil - 1}) \oplus (y_i \oplus \cdots \oplus y_n) \qquad i > \lceil n/2 \rceil$$

A linear family with feeble one-wayness of order $\frac{3}{2}$

Theorem

For all $n \geq 5$, the functions ϕ_n satisfy

$$C(\phi_n) = n + 1 \quad \textit{and} \quad C(\phi_n^{-1}) = \lfloor \frac{3}{2}(n-1) \rfloor$$

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- *It can be easily verified that the previous two bounds are exact.*

Nonlinear family with feeble one-wayness of order 2

Remark

It is easy to modify the previous family to make it one-way of order 2 (still being linear). However, it is even simipler to construct a non-linear family.

Construction

The family ν_n results from composition $\beta_n(\alpha_n(x))$ of linear permutation $\alpha_n([x_1, \ldots, x_n]) = (z_1, \ldots, z_n)$ with a nonlinear permutation $\beta_n([z_1, \ldots, z_n]) = (y_1, \ldots, y_n)$, where:

$$z_i(x) = x_i \oplus x_{i+1} \quad \text{for i} \neq n; \quad z_n(x) = x_n$$

$$y_i(z) = z_i \quad \text{for i} \neq n; \quad y_n(z) = z_n \oplus [\overline{(z_1 \oplus \cdots \oplus z_{n-2})} \wedge z_{n-1}]$$

Nonlinear family with feeble one-wayness of order 2

Construction

The inverse permutations $\beta_n^{-1}([y_1, \ldots, y_n]) = (z_1, \ldots, z_n)$ and $\alpha_n^{-1}([z_1, \ldots, z_n]) = (x_1, \ldots, x_n)$ will be:

$$z_i(y) = y_i \quad \text{for i} \neq n; \quad z_n(y) = y_n \oplus [\overline{(y_1 \oplus \cdots \oplus y_{n-2})} \wedge y_{n-1}]$$

$$x_i(z) = z_i \oplus \cdots \oplus z_n \quad \text{for i} \neq n; \quad x_n(z) = z_n$$

Nonlinear family with feeble one-wayness of order 2

Construction

The composition of α_n and β_n yields $\nu_n(x) = \beta_n(\alpha_n(x))[y_1, \dots, y_n]$, and $\nu_n^{-1}(y) = \alpha_n^{-1}(\beta_n^{-1}(y)) = [x_1, \dots, x_n]$ where:

$$y_i(x) = x_i \oplus x_{i+1} \quad \text{for i} \neq n; \quad y_n(x) = x_n \oplus [\overline{(x_1 \oplus x_{n-1})} \wedge (x_{n-1} \oplus x_n)]$$

$$x_i(y) = (y_i \oplus \cdots \oplus y_n) \oplus [\overline{(y_1 \oplus \cdots \oplus y_{n-2})} \wedge y_{n-1}] \quad \text{for i} \neq n$$

$$x_n(y) = y_n \oplus [\overline{(y_1 \oplus \cdots \oplus y_{n-2})} \wedge y_{n-1}]$$

Theorem

For all $n \geq 4$, the functions ν_n satisfy

$$C(\nu_n) = n + 2 \quad \textit{and} \quad C(\nu_n^{-1}) = 2(n-1)$$

Conclusion

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Conclusion

- The results described in the first two sections of this talk were obtained together with Alexander S. Kulikov and Arist Kojevnikov.
- Now we are working on improving the results of the last section (obtained by Alain Hiltgen) together with my advisor Edward A. Hirsch.
- It is not easy to improve the constant 2 in the last section, because you need to prove a nontrivial lower bound to do this.

Thank you for your attention!