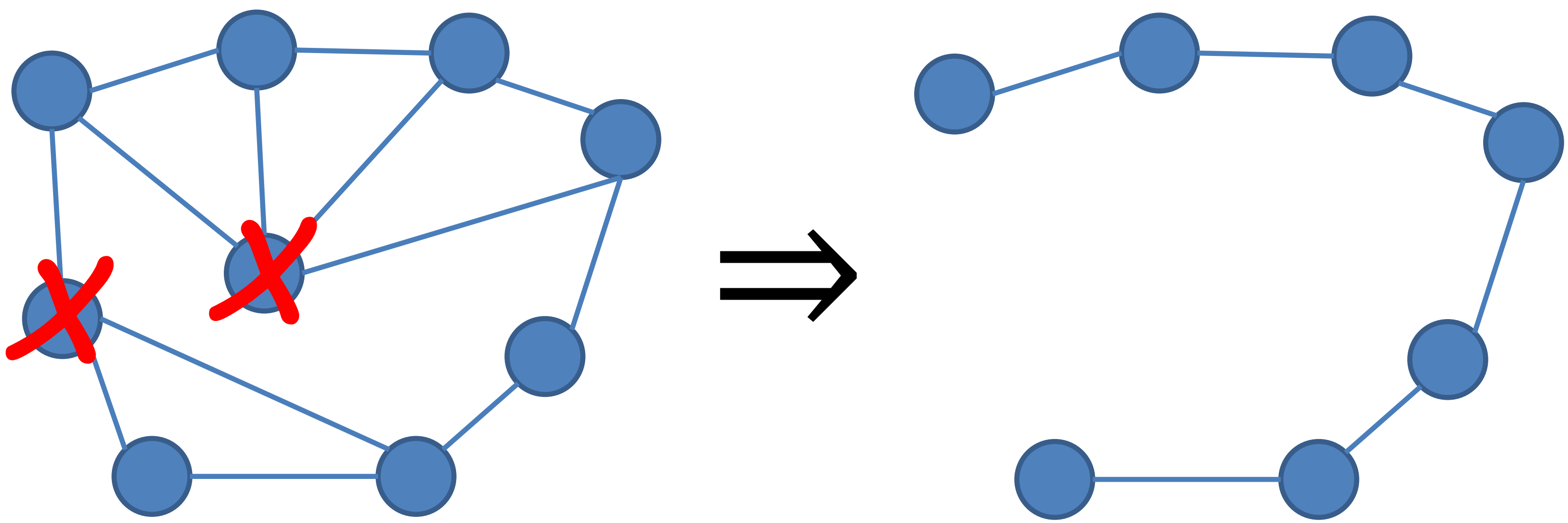
Primal-dual algorithms for **node-weighted** network design in **planar graphs**

Grigory Yaroslavtsev
Penn State
(joint work with Piotr Berman)

Feedback Vertex Set Problems

- **Given:** a collection of **cycles** in a graph
- **Goal:** break them, removing a small # of **vertices**

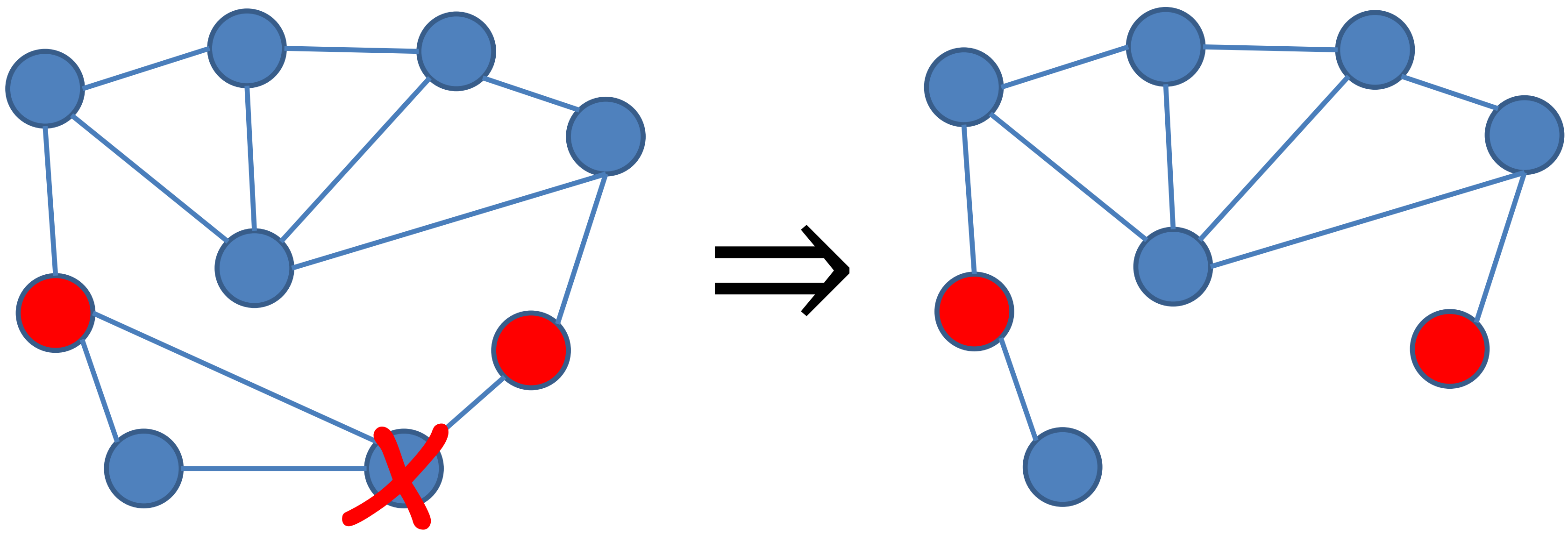
Example: Collection = All cycles



Weighted vertices => remove set of min cost

FVS: Flavors and toppings

- All cycles = Feedback Vertex Set
- All Directed cycles = **Directed** FVS
- All **odd-length** cycles = Bipartization
- Cycles through a **subset** of vertices = **Subset** FVS



FVS in general graphs

- **NP-hard** (even in planar graph [Yannakakis])

Problem	Approximation
FVS	**2** [Becker, Geiger; Bafna, Berman, Fujito]
Bipartization	$\boldsymbol{O}(\log \boldsymbol{n})$ [Garg, Vazirani, Yannakakis]
Directed FVS	$\boldsymbol{O}(\log \boldsymbol{n} \log\log \boldsymbol{n})$ [Even, Naor, Schieber, Sudan]
Subset FVS	**8** [Even, Naor, Zosin]

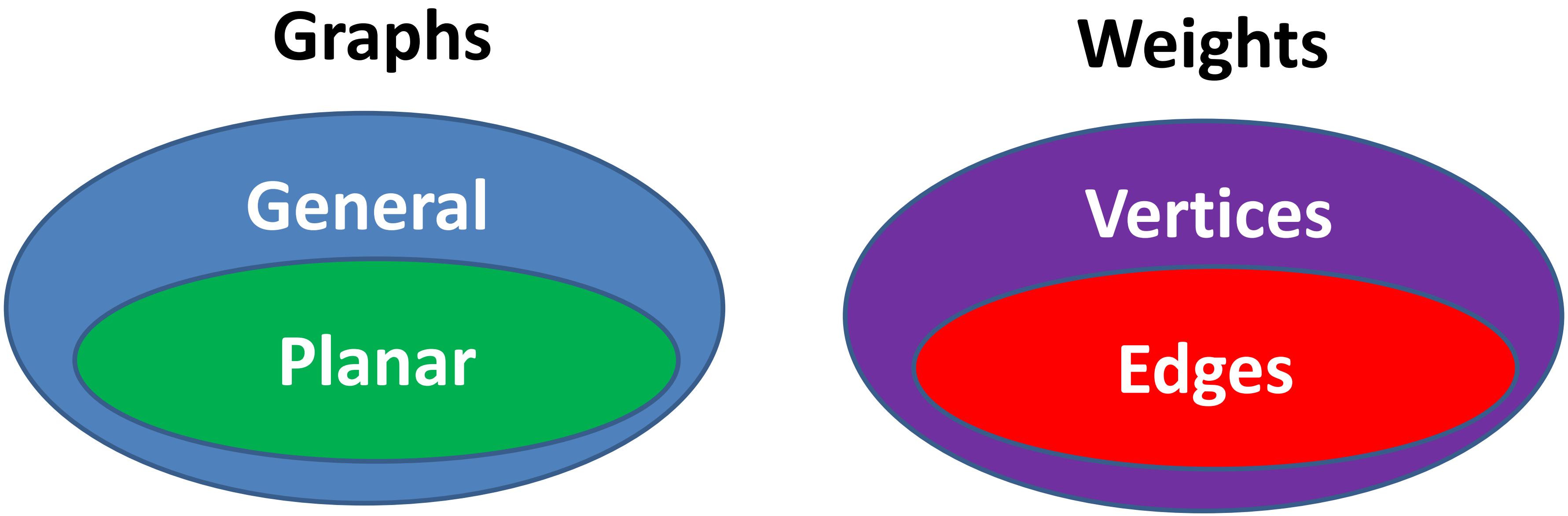
- **MAX-SNP complete** [Lewis, Yannakakis; Papadimitriou, Yannakakis] **=>**
  - 1.3606, if $\mathrm{P} \neq NP$ [Dinur, Safra]
  - $2-\epsilon$ under UGC [Khot, Regev]

FVS in planar graphs (via primal-dual)

- **NP-hard** (even in planar graph [Yannakakis])

<table>
<tr><th>Problems</th><th colspan="2">Previous work</th><th>This work</th></tr>
<tr><td>FVS</td><td>10
[Bar-Yehuda, Geiger, Naor, Roth]</td><td rowspan="2">3
[Goemans, Williamson, 96]</td><td rowspan="4">2.4 (2.57)</td></tr>
<tr><td>BIP, D-FVS, S-FVS</td><td></td></tr>
<tr><td>Node-Weighted Steiner Forest</td><td rowspan="2">6
[Demaine, Hajiaghayi, Klein’09]</td><td>3
[Moldenhauer’11]</td></tr>
<tr><td>More general class of problems</td><td></td></tr>
</table>

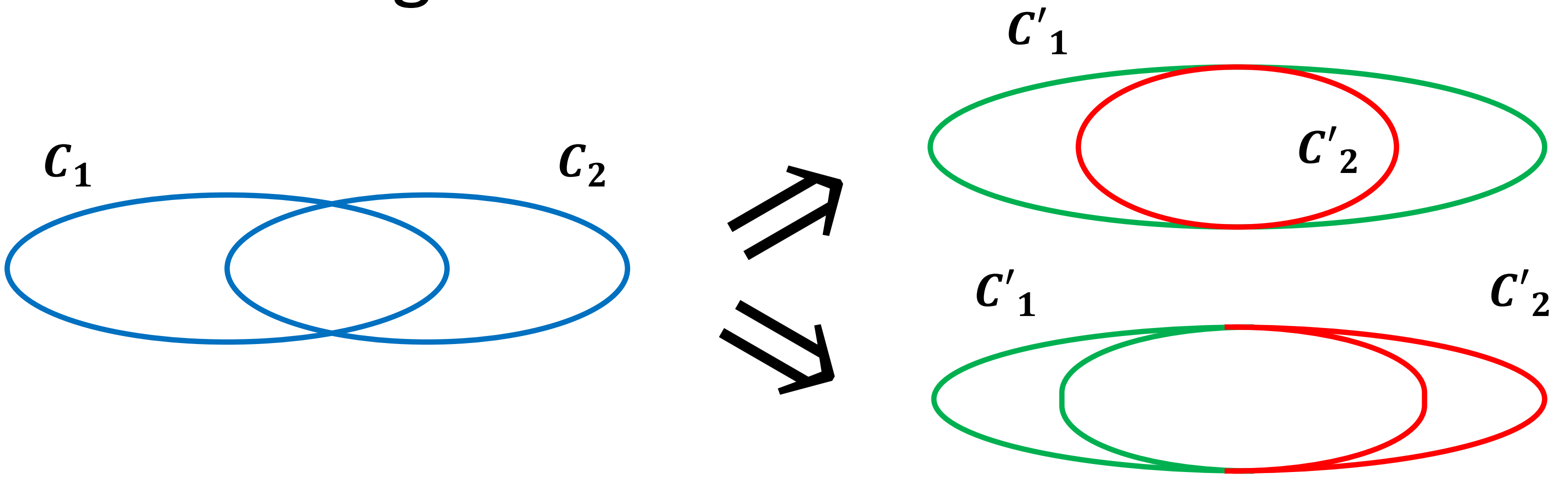
Bigger picture



- **Feedback Edge Set** in **general graphs** = Complement of MST
- **Planar edge-weighted BIP** and **D-FVS** are also in P
- **Planar edge-weighted Steiner Forest** has a PTAS [Bateni, Hajiaghayi, Marx, STOC'11]
- **Planar unweighted Feedback Vertex Set** has a PTAS [Baker; Demaine, Hajiaghayi, SODA'05]

Class 1: Uncrossing property

- Uncrossing:



- Uncrossing property of a family of cycles C:

> For every two crossing cycles $\boldsymbol{C_1}, \boldsymbol{C_2} \in C$, one of their two uncrossings has $\boldsymbol{C'_1}, \boldsymbol{C'_2} \in C$.

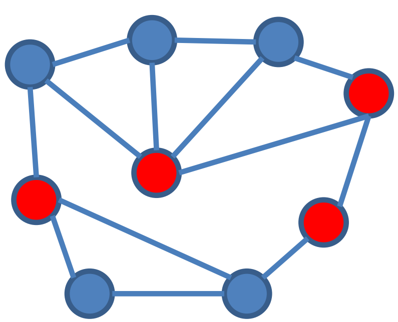
- Holds for all FVS problems, crucial for the algorithm of GW

Proper functions [GW, DHK]

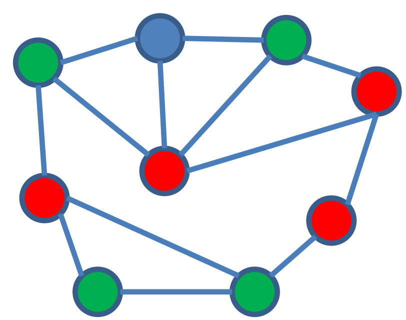
- A function $f: 2^V \to \{0,1\}$ is **proper** if $f(\emptyset) = 0$,
  - **Symmetry:** $f(S) = f(V \setminus S)$
  - **Disjointness:** If $S_1 \cap S_2 = \emptyset$ and $f(S_1) = f(S_2) = 0$ **=>** $f(S_1 \cup S_2) = 0$

- A set $S \subseteq V$ is **active**, if $f(S) = 1$
- Boundary $\Gamma(S)$:

S



$\Gamma(S)$



- A boundary $\Gamma(S) \subseteq V$ is **active**, if S is **active**

Class 2: Hitting set IP [DHK]

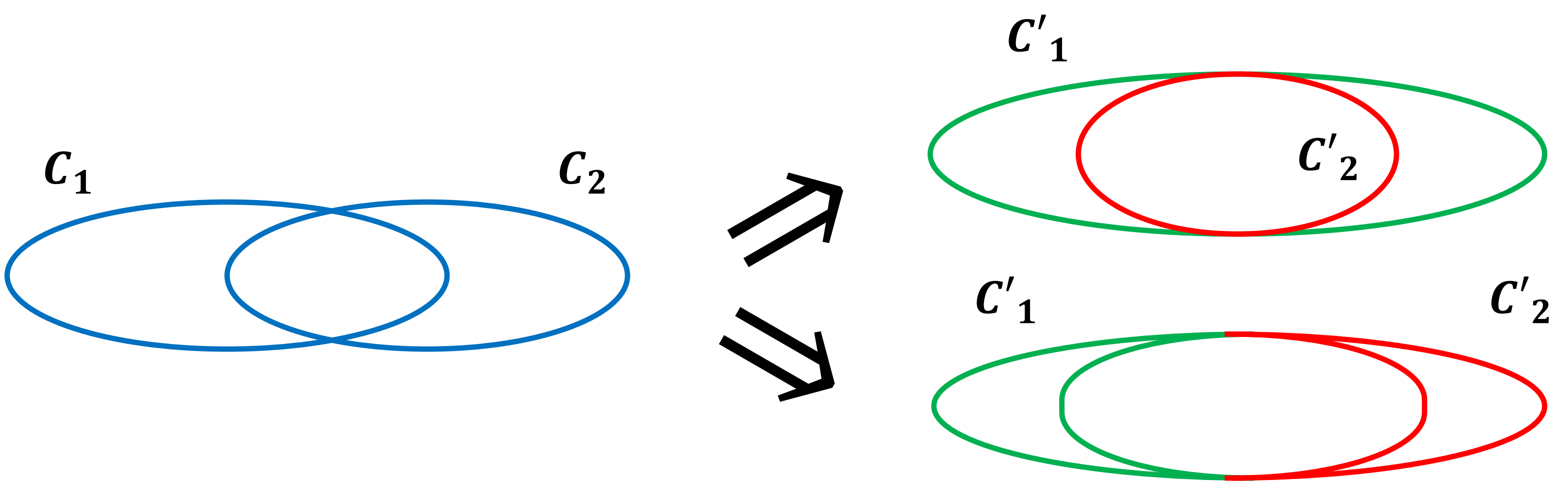
- **The class of problems:**

  Minimize: $\sum_{v \in V} w(v)x(v)$

  Subject to: $\sum_{v \in \Gamma(S)} x(v) \geq f(S)$, for all $S \subseteq V$

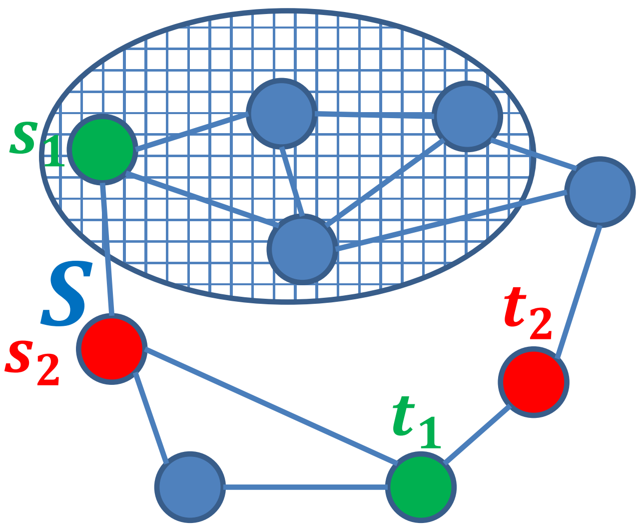
  $x_v \in \{0,1\}$,

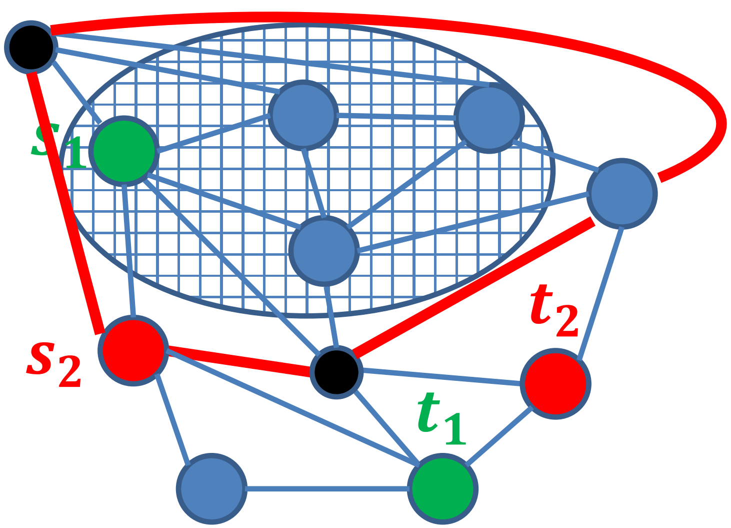
  where f is a **proper function**
- **Theorem:** f is proper => the collection of all active boundaries forms an uncrossable family (requires triangulation)
- Proof sketch: f is proper => in one of the cases both interior sets are active => their boundaries are active



Class 1 = Class 2

- **Example:** Node-Weighted Steiner Forest
  - Connect pairs (s_i, t_i) via a subset of nodes of min cost
  - Proper function $f(S) = 1$ iff $|S \cap \{s_i, t_i\}| = 1$ for some i.



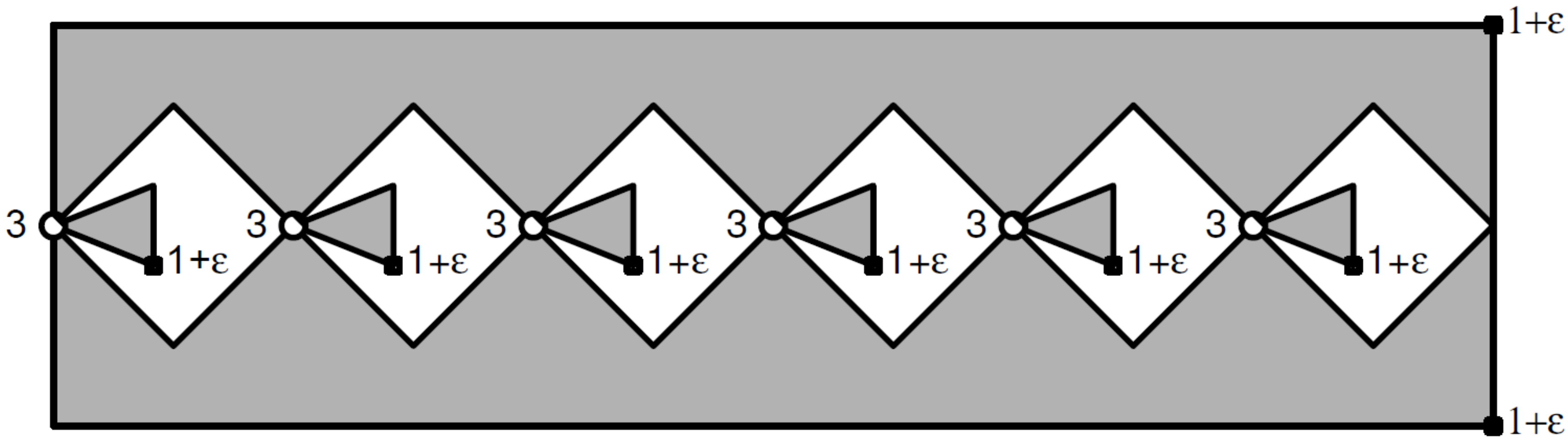


Primal-dual method (local-ratio version)

- Given: G (graph), w (weights), $\boldsymbol{C}$ (cycles)
  - $\bar{\boldsymbol{w}} = w$
  - $\boldsymbol{S}$ = set of all vertices of zero weight
  - While $\boldsymbol{S}$ is not a hitting set for $\boldsymbol{C}$
    - $\boldsymbol{M}$ = collection of cycles returned by oracle **Violation** (G, **C**, $\boldsymbol{S}$)
    - $\boldsymbol{c_M}(u)$ = # of cycles in **M**, which contain u
    - $\bar{\boldsymbol{w}}(\boldsymbol{u}) = \bar{\boldsymbol{w}}(\boldsymbol{u}) - \min_{u \in V \setminus S} \frac{\bar{\boldsymbol{w}}(\boldsymbol{u})}{\boldsymbol{c_M}(u)} \cdot \boldsymbol{c_M}(u)$
    - $\boldsymbol{S}$ = set of all vertices of zero weight $\bar{\boldsymbol{w}}$
  - **Return** a minimal hitting set $H \subset \boldsymbol{S}$ for $\boldsymbol{C}$

Oracle 1 = Face-minimal cycles [GW]

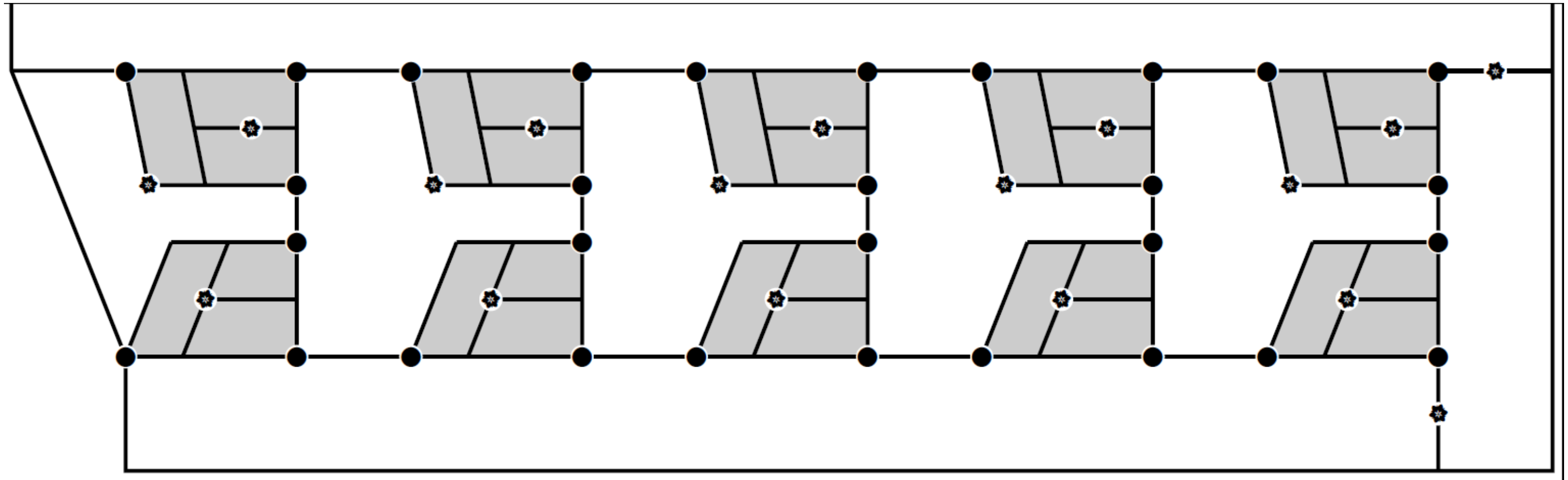
- Example for Subset FVS with $\gamma = 3$:



- Oracle returns all gray cycles => all white nodes are selected
- Cost = 3 * # blocks, OPT $\sim (1+\epsilon)$ * # blocks

Oracle 2 = Pocket removal [GW]

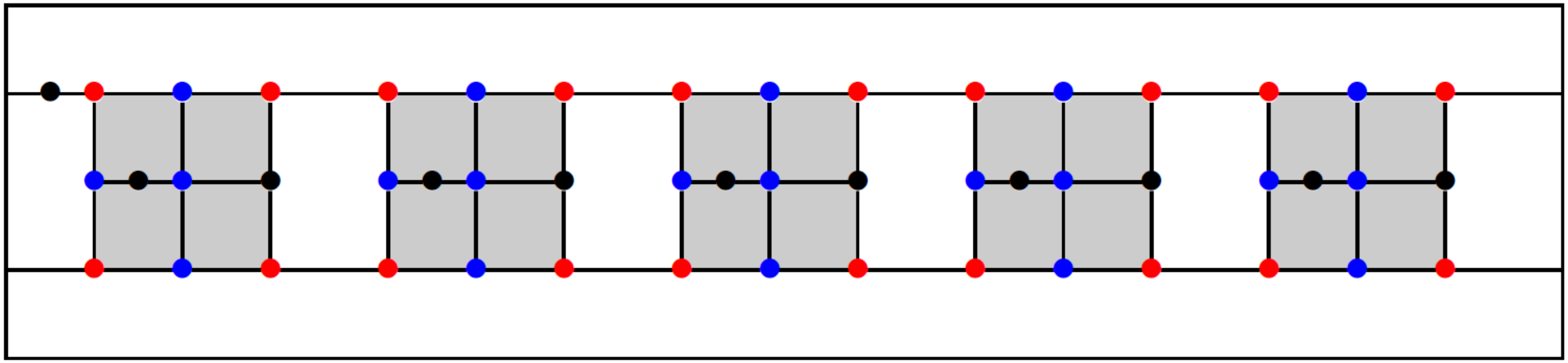
- **Pocket** defined by two cycles: region between their common points containing another cycle
- **New oracle**: no pocket => all face-minimal cycles, otherwise run recursively inside any pocket.
- **Our analysis:** $\gamma = \frac{18}{7} \approx 2.57$



Each ● costs 3, each ✿ costs degree+ε

Oracle 3 = Triple pocket removal

- **Triple pocket =** region defined by three cycles
- **Analysis:** $\boldsymbol{\gamma = 2.4}$



Red nodes have cost 3, other nodes have cost degree+ε, black nodes form the optimum solution

Open problems

For our class of node-weighted problems:

- **Big question:** APX-hardness or a PTAS?
- **Integrality gap** = 2, how to approach it?
  - Pockets of higher multiplicities are harder to analyze
  - Pockets cannot go beyond $\mathbf{2 + \delta}$

Applications and ramifications

- **Applications:** from maintenance of power networks to computational sustainability
- **Example:** VLSI design.
  - Primal-dual approximation algorithms of Goemans and Williamson are competitive with heuristics [Kahng, Vaya, Zelikovsky]
- Connections with bounds on the size of FVS
  - Conjectures of Akiyama and Watanabe and Gallai and Younger [see GW for more details]

Approximation factor

- **Theorem [GW'96]:** If for any minimal solution **H** the set ***M*** returned by the oracle satisfies:

$$\sum_{u \in H} c_M(u) \leq \gamma |M|,$$

  then the primal-dual algorithm has approximation γ.
- **Examples of oracles:**
  - **Single cycle:** $\gamma \leq 10$ [Bar-Yehuda, Geiger, Naor, Roth]
  - **Single cycle:** $\gamma \leq 5$ [Goemans, Williamson]
  - **Collection of all face-minimal cycles:** $\gamma \leq 3$ [Goemans, Williamson]