Motivation for Sublinear-Time Algorithms

Massive datasets

- world-wide web
- online social networks
- genome project
- sales logs
- census data
- high-resolution images
- scientific measurements

Long access time

- communication bottleneck (slow connection)
- implicit data (an experiment per data point)

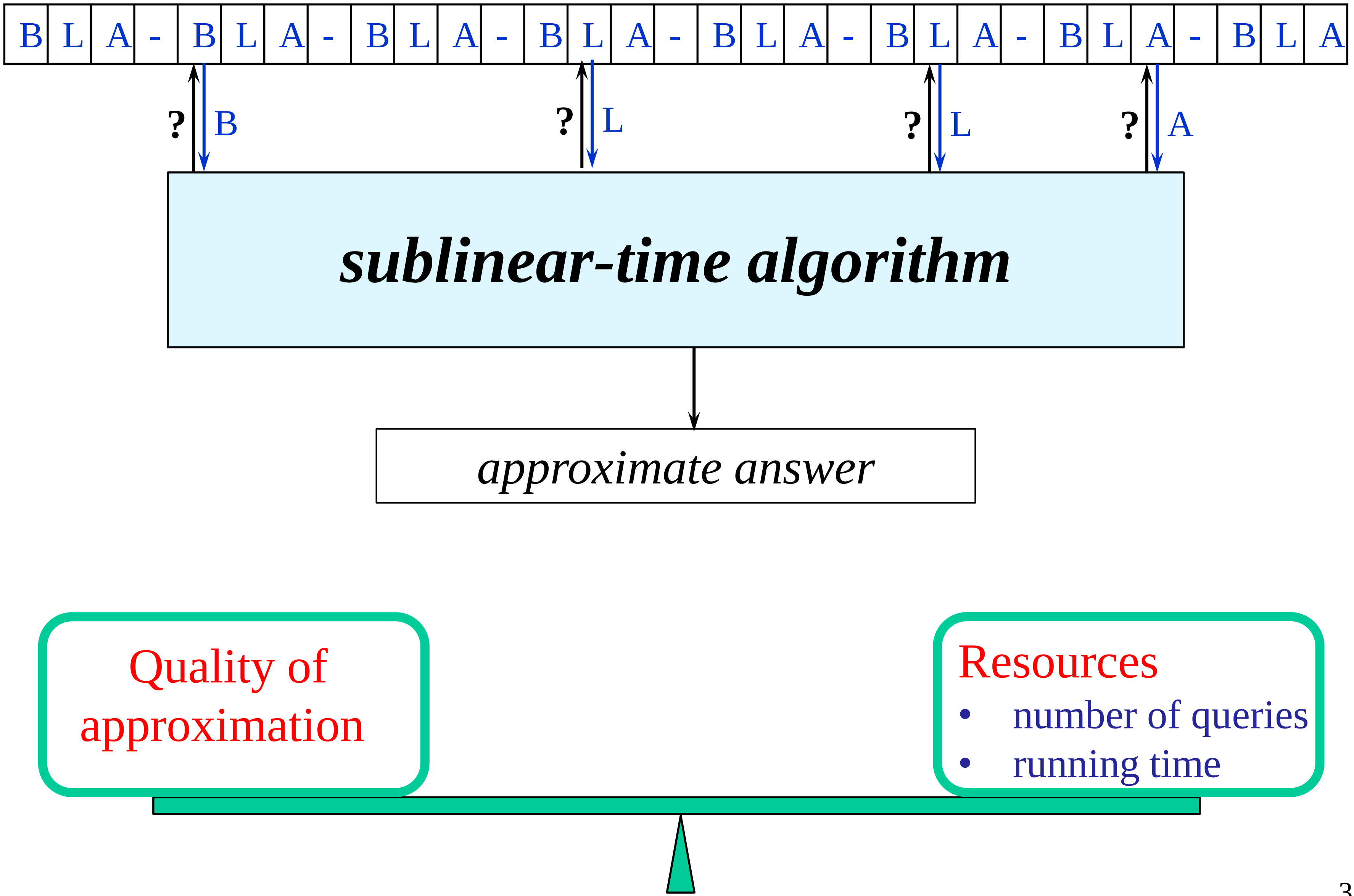
What Can We Hope For?

- What can an algorithm compute if it
  - reads only a **sublinear** portion of the data?
  - runs in **sublinear** time?

- Some problems have exact deterministic solutions

- For most interesting problems algorithms must be
  - approximate
  - randomized

A Sublinear-Time Algorithm



Types of Approximation

Classical approximation

- need to compute a value
  - output should be close to the desired value
  - example: average

Property testing

- need to answer YES or NO
  - Intuition: only require correct answers on two sets of instances that are very different from each other

Classical Approximation

A Simple Example

Approximate Diameter of a Point Set [Indyk]

Input: m points, described by a distance matrix D

- D_{ij} is the distance between points i and j
- D satisfies triangle inequality and symmetry

(Note: input size is $n = m^2$)

Let i, j be indices that maximize D_{ij}.

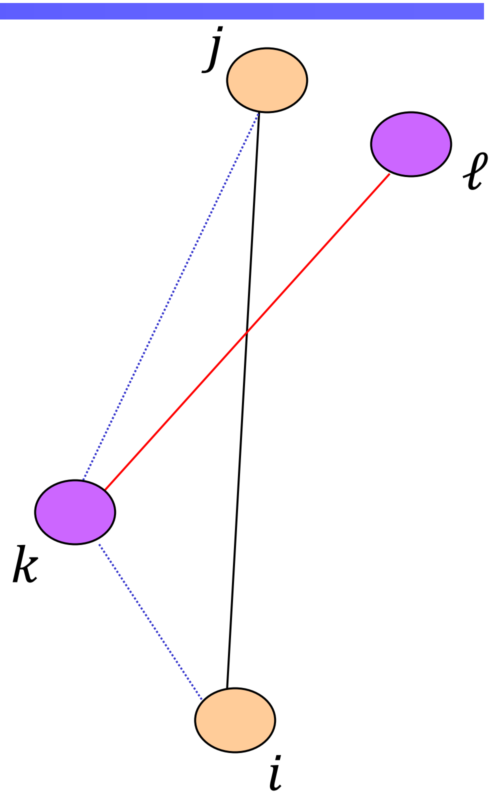
Maximum D_{ij} is the *diameter*.

- Output: (k, ℓ) such that $D_{k\ell} \geq D_{ij}/2$

Algorithm and Analysis

Algorithm (m, D)

1. Pick k arbitrarily
2. Pick ℓ to maximize $D_{k\ell}$
3. Output (k, ℓ)



- Approximation guarantee

$$D_{ij} \le D_{ik} + D_{kj} \text{ (triangle inequality)}$$
$$\le D_{k\ell} + D_{k\ell} \text{ (choice of } \ell \text{ + symmetry of } D\text{)}$$
$$\le 2Dk_{\ell}$$

- Running time: $O(m) = O(m = \sqrt{n})$

A rare example of a deterministic sublinear-time algorithm

Property Testing

Property Testing: YES/NO Questions

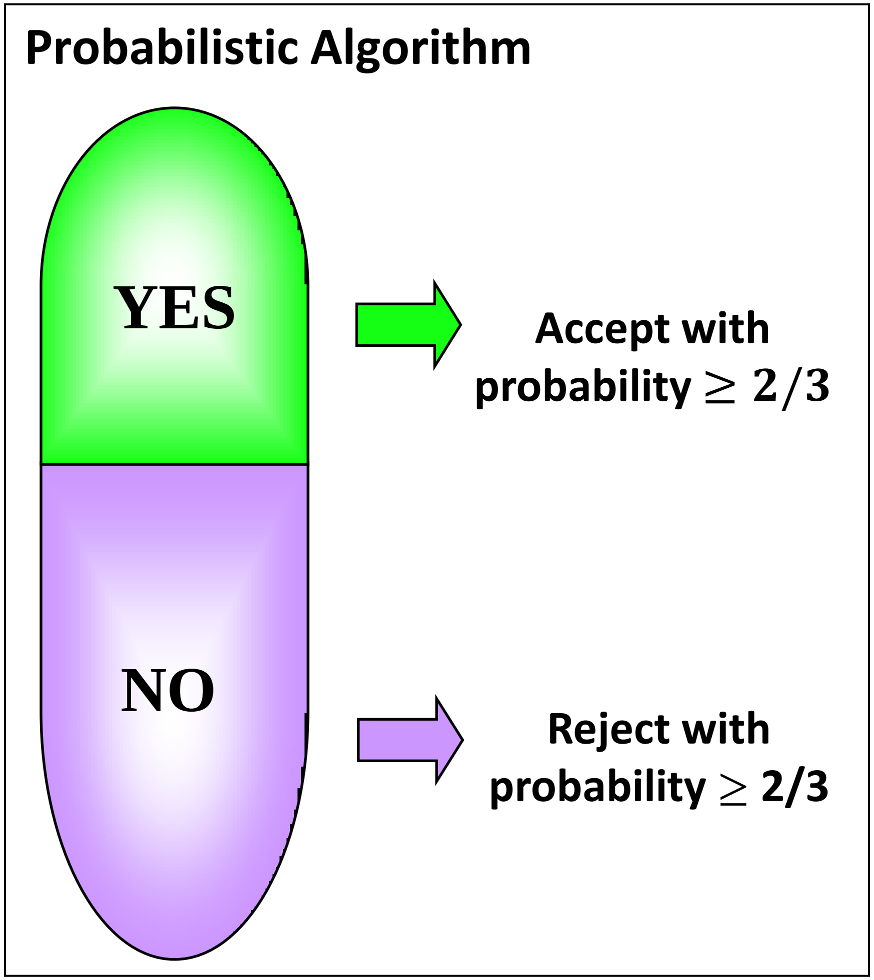
Does the input satisfy some property? (YES/NO)

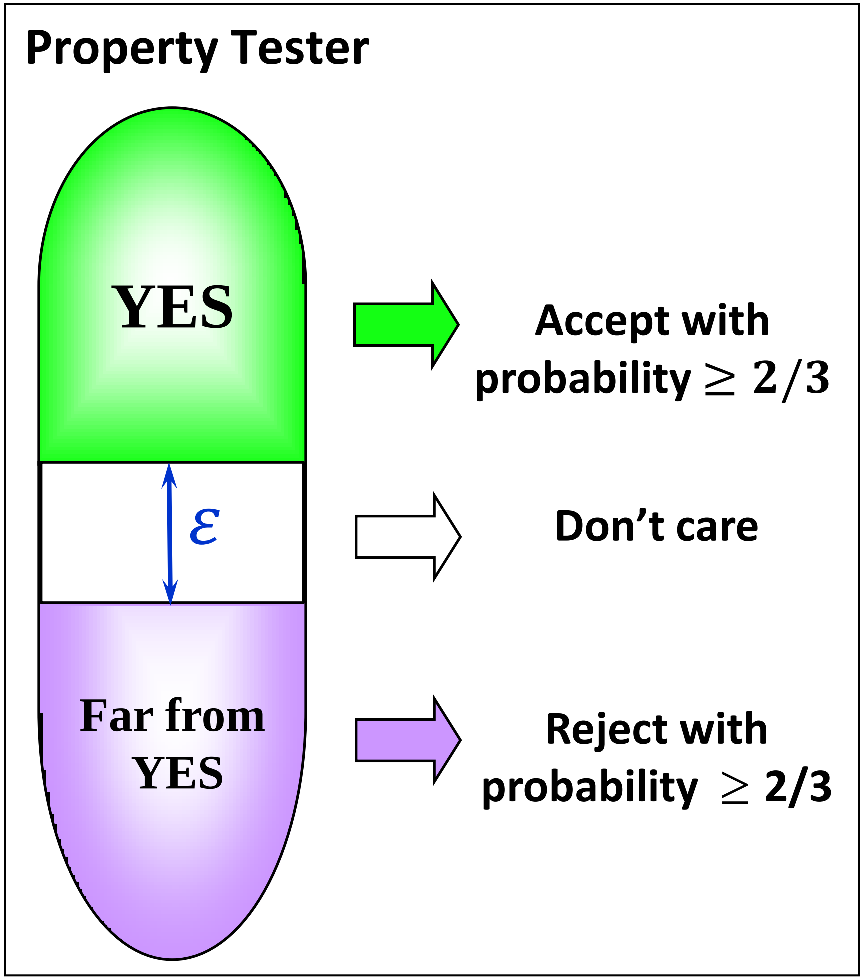
"in the ballpark" vs. "out of the ballpark"

Does the input satisfy the property
or is it far from satisfying it?

- sometimes it is the right question (probabilistically checkable proofs (PCPs))
- as good when the data is constantly changing (WWW)
- fast sanity check to rule out inappropriate inputs (airport security questioning)

Property Tester Definition





ε-far = differs in many places ($\geq \varepsilon$ fraction of places)

Randomized Sublinear Algorithms

Toy Examples

Property Testing: a Toy Example

0	0	0	1	…	0	1	0	0

Input: a string $w \in \{0,1\}^n$

Question: Is $w = 00 \ldots 0$?

Requires reading entire input.

Approximate version: Is $w = 00 \ldots 0$ or does it have $\geq \varepsilon n$ 1's ("errors")?

Test (n, w)

1. Sample $s = 2/\varepsilon$ positions uniformly and independently at random
2. If 1 is found, **reject**; otherwise, **accept**

Analysis: If $w = 00 \ldots 0$, it is always accepted.

Used: $1 - x \leq e^{-x}$

If w is ε-far, Pr[error] = Pr[no 1's in the sample]$\leq (1-\varepsilon)^s \leq e^{-\varepsilon s} = e^{-2} < \frac{1}{3}$

Witness Lemma

If a test catches a witness with probability $\geq p$,
then $s = \frac{2}{p}$ iterations of the test catch a witness with probability $\geq 2/3$.

Randomized Approximation: a Toy Example

0	0	0	1	…	0	1	0	0

Input: a string $w \in \{0,1\}^n$

Goal: Estimate the fraction of 1's in w (like in polls)

It suffices to sample $s = 1 / \varepsilon^2$ positions and output the average
to get the fraction of 1's $\pm\varepsilon$ (i.e., additive error ε) with probability $\geq$ 2/3

Hoeffding Bound

Let $\mathrm{Y}_1, \ldots, \mathrm{Y}_s$ be independently distributed random variables in [0,1] and let $\mathrm{Y} = \sum_{i=1}^{s} \mathrm{Y}_i$ (sample sum). Then $\Pr[|\mathrm{Y} - \mathrm{E}[\mathrm{Y}]| \geq \delta] \leq 2\mathrm{e}^{-2\delta^2/s}$.

Y_i = value of sample i. Then E[Y] $= \sum_{i=1}^{s} \mathrm{E}[\mathrm{Y}_i] = s \cdot$ (fraction of 1's in w)

$$\Pr[|(\text{sample average}) - (\text{fraction of } 1\text{'s in } w)| \geq \varepsilon] = \Pr[|\mathrm{Y} - \mathrm{E}[\mathrm{Y}]| \geq \varepsilon s]$$
$$\leq 2\mathrm{e}^{-2\delta^2/s} = 2e^{-2} < 1/3$$

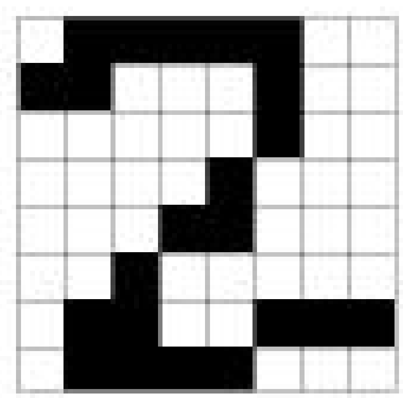
Apply Hoeffding Bound with $\delta = \varepsilon s$

substitute $s = 1 / \varepsilon^2$

Property Testing

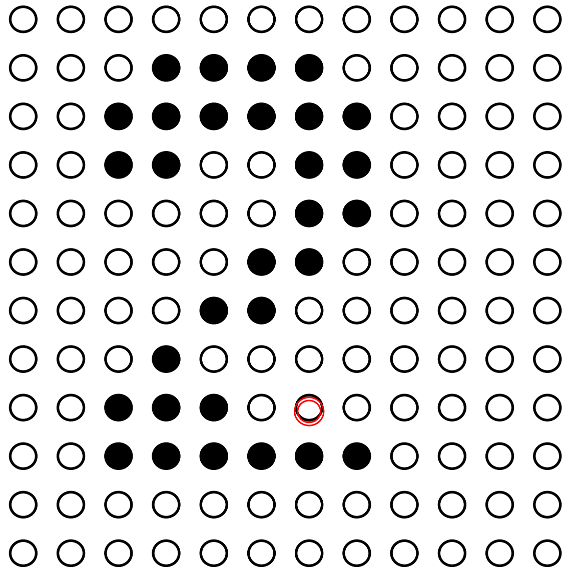
Simple Examples

Testing Properties of Images



Pixel Model

Input: $n \times n$ matrix of pixels
(0/1 values for black-and-white pictures)



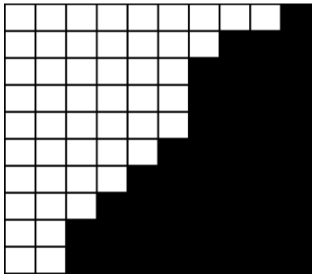
Query: point (i_1, i_2)

Answer: color of (i_1, i_2)

Testing if an Image is a Half-plane [R03]

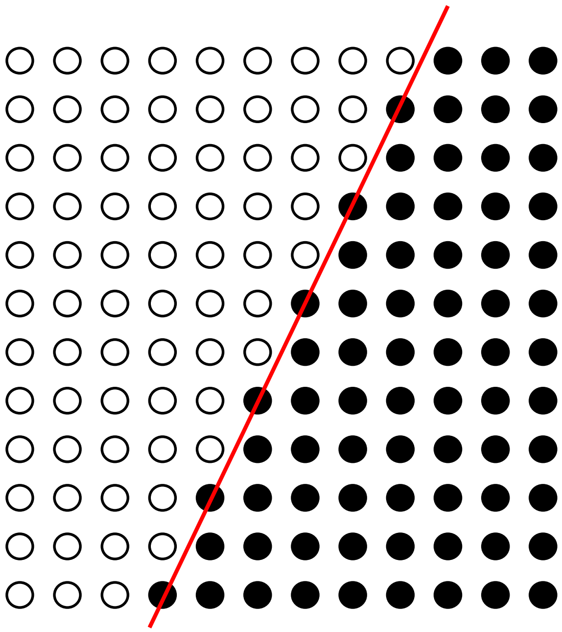
A half-plane or

ε-far from a half-plane?

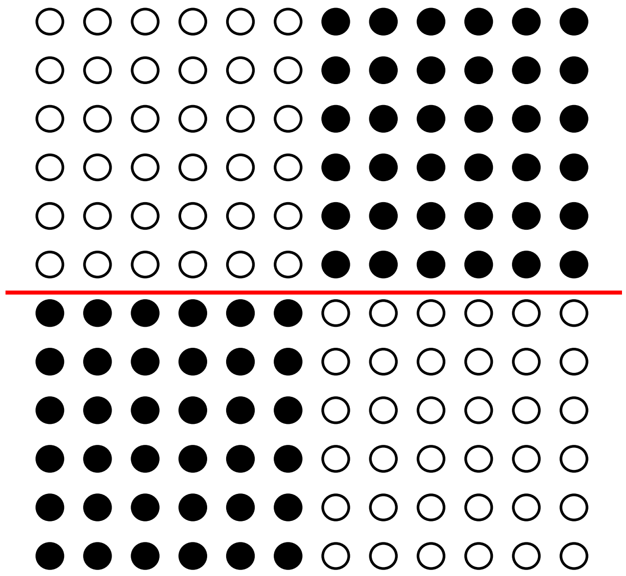


O(1/ε) time

Half-plane Instances

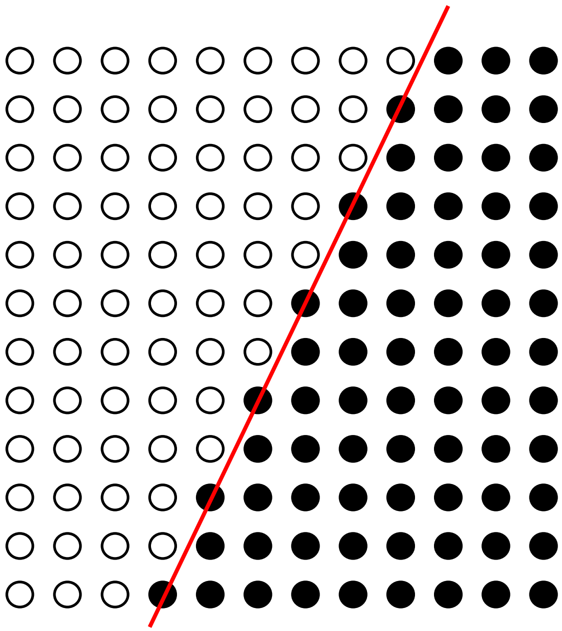


A half-plane

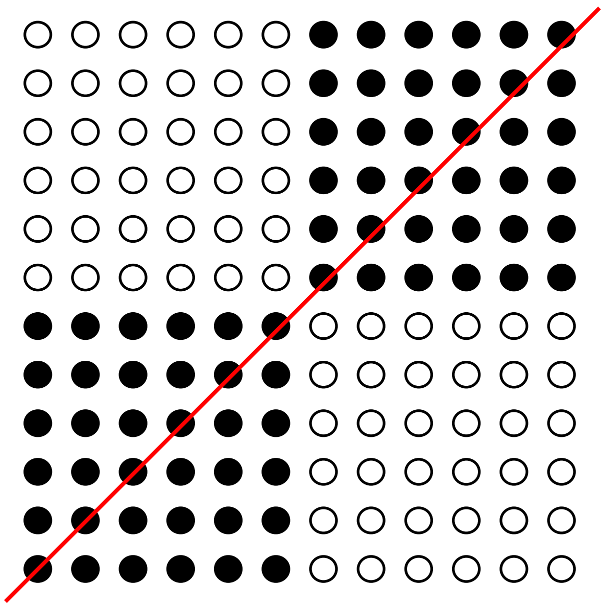


$\frac{1}{4}$-far from a half-plane

Half-plane Instances

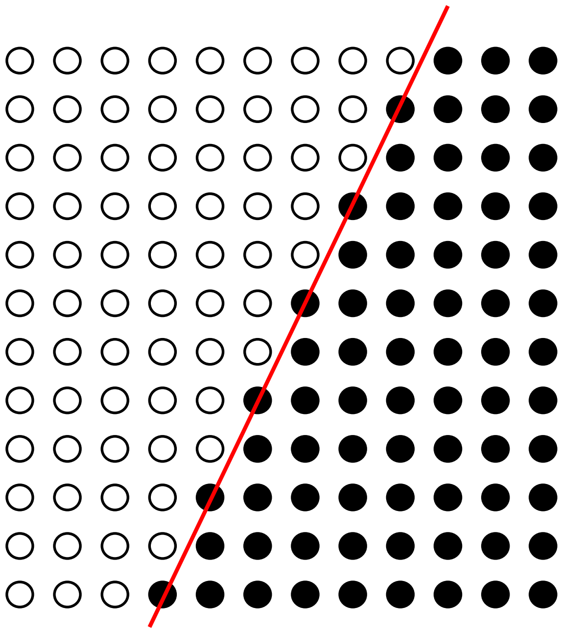


A half-plane

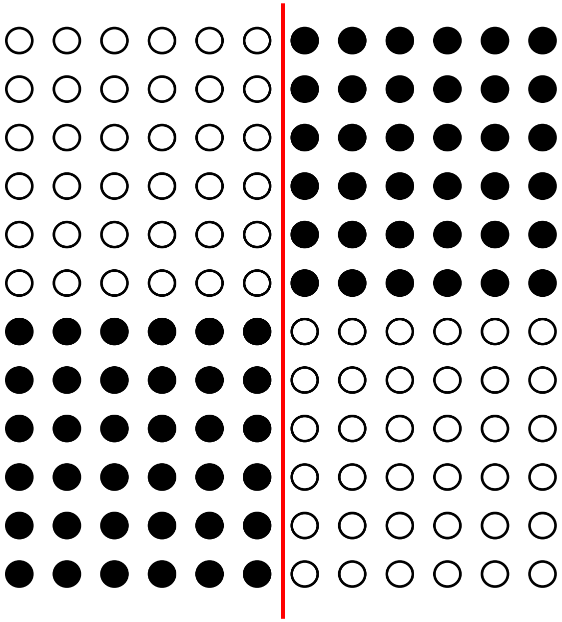


$\frac{1}{4}$-far from a half-plane

Half-plane Instances

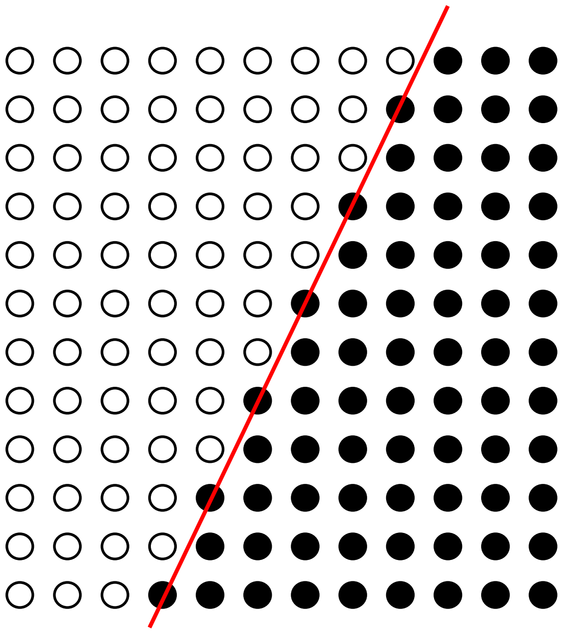


A half-plane

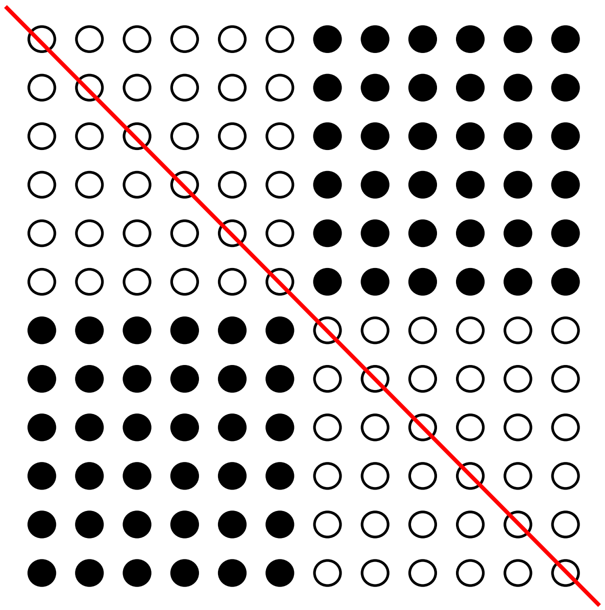


$\frac{1}{4}$-far from a half-plane

Half-plane Instances

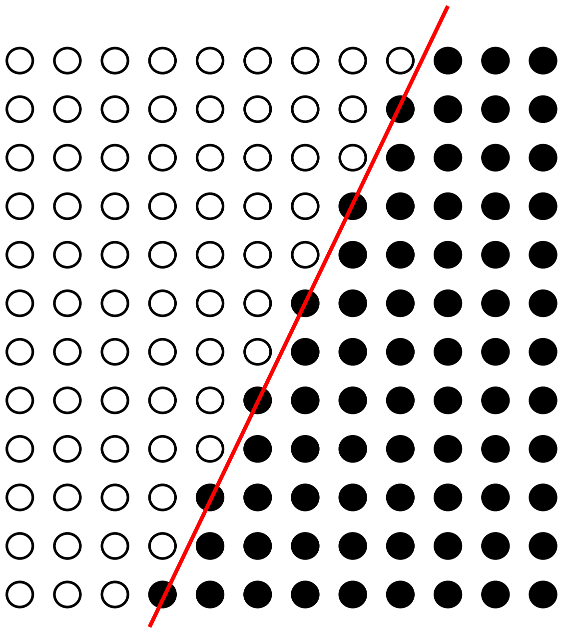


A half-plane

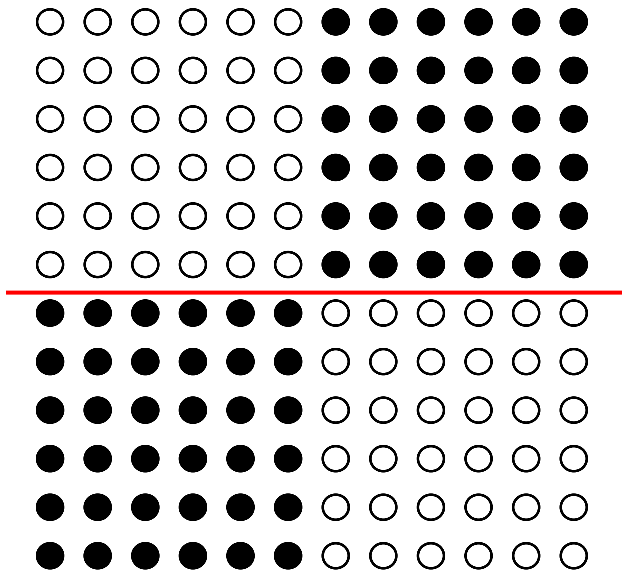


$\frac{1}{4}$-far from a half-plane

Half-plane Instances

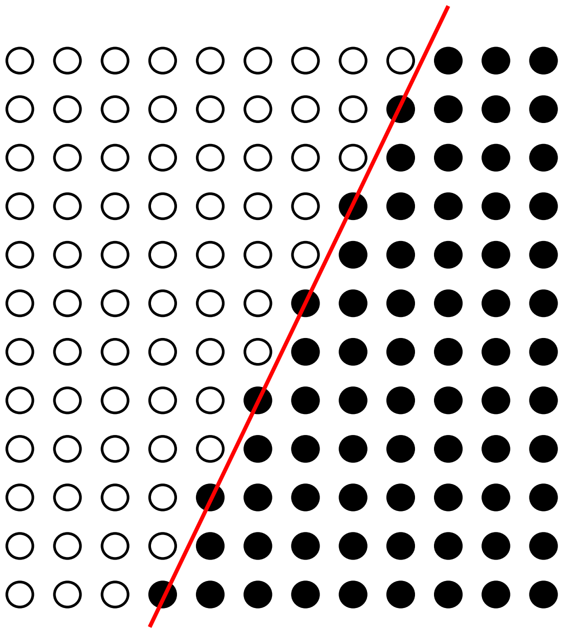


A half-plane

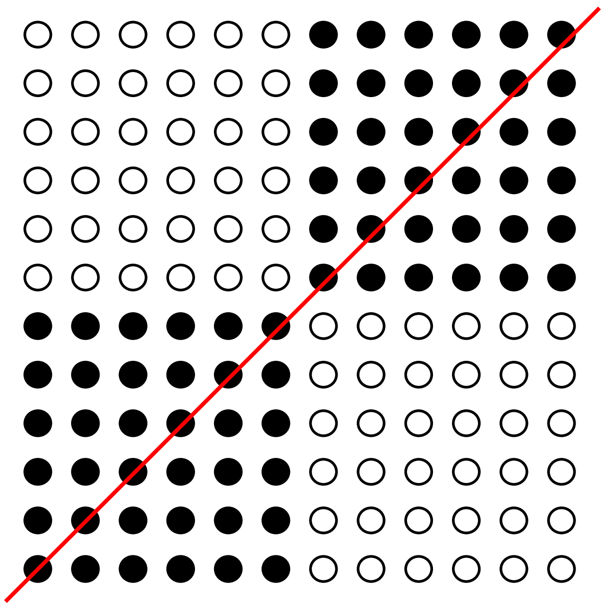


$\frac{1}{4}$-far from a half-plane

Half-plane Instances

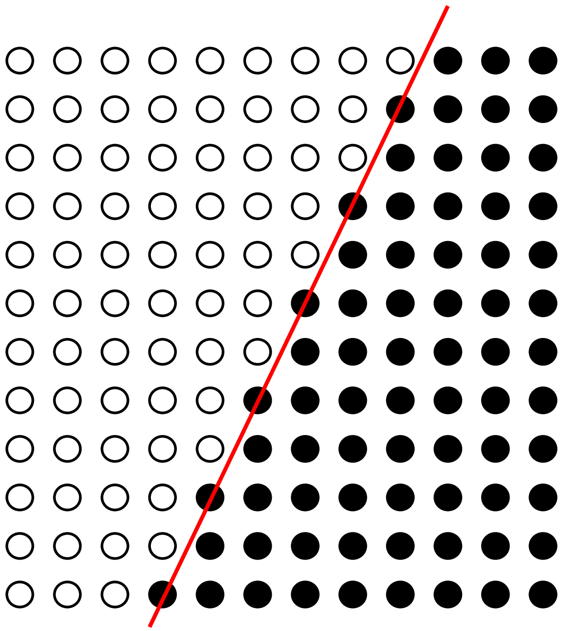


A half-plane

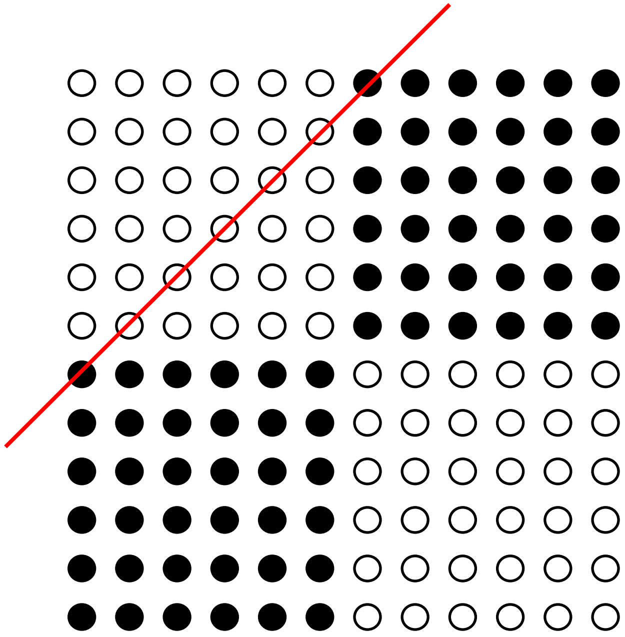


$\frac{1}{4}$-far from a half-plane

Half-plane Instances



A half-plane



$\frac{1}{4}$-far from a half-plane

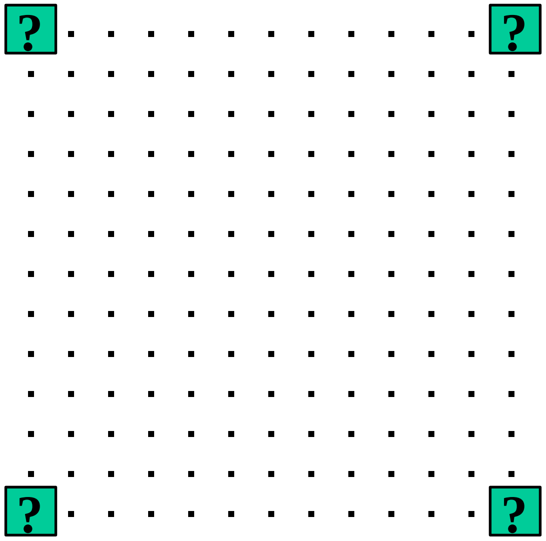
Strategy

"Testing by implicit learning" paradigm

- Learn the outline of the image by querying a few pixels.
- Test if the image conforms to the outline by random sampling, and reject if something is wrong.

Half-plane Test

Claim. The number of sides with different corners is 0, 2, or 4.

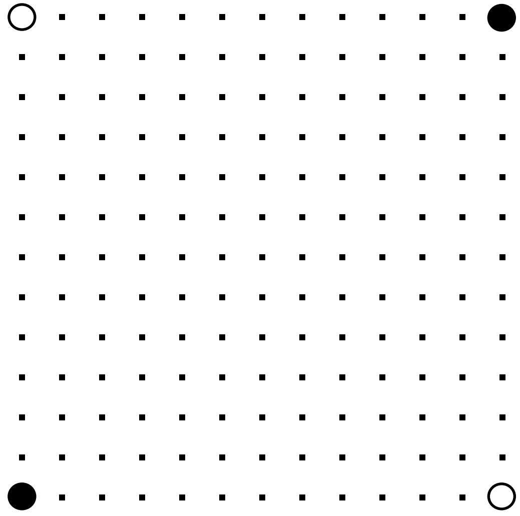


Algorithm

1. Query the corners.

Half-plane Test: 4 Bi-colored Sides

Claim. The number of sides with different corners is 0, 2, or 4.



Analysis

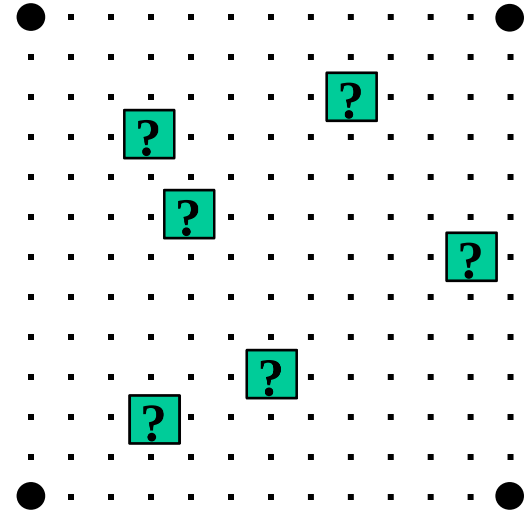
- If it is 4, the image cannot be a half-plane.

Algorithm

1. Query the corners.
2. If the number of sides with different corners is 4, **reject**.

Half-plane Test: 0 Bi-colored Sides

Claim. The number of sides with different corners is 0, 2, or 4.



Analysis

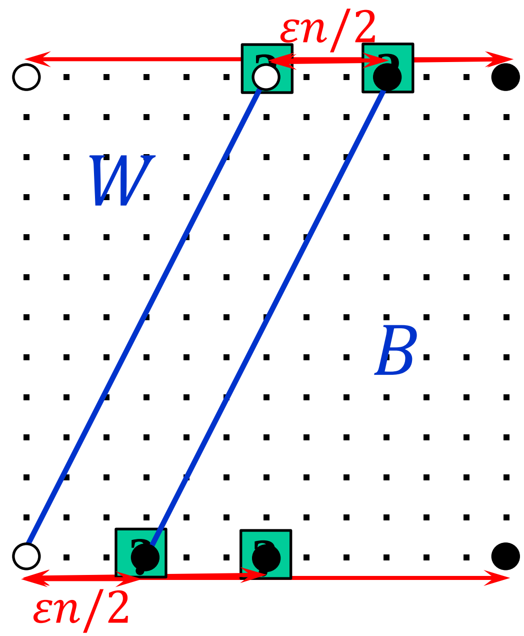
- If all corners have the same color, the image is a half-plane if and only if it is unicolored.

Algorithm

1. Query the corners.
2. If all corners have the same color c, test if all pixels have color c (as in Toy Example 1).

Half-plane Test: 2 Bi-colored Sides

Claim. The number of sides with different corners is 0, 2, or 4.



Analysis

- The area outside of $W \cup B$ has $\leq \varepsilon n^2/2$ pixels.
- If the image is a half-plane, W contains only white pixels and B contains only black pixels.
- If the image is ε-far from half-planes, it has $\geq \varepsilon n^2/2$ wrong pixels in $W \cup B$.
- By Witness Lemma, $4/\varepsilon$ samples suffice to catch a wrong pixel.

Algorithm

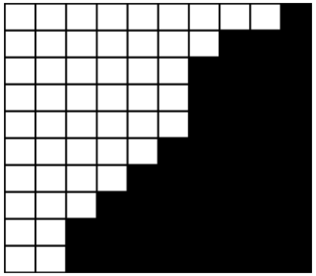
1. Query the corners.
2. If # of sides with different corners is 2, on both sides find 2 different pixels within distance $\varepsilon n/2$ by binary search.
3. Query $4/\varepsilon$ pixels from $W \cup B$
4. **Accept** iff all W pixels are white and all B pixels are black.

Testing if an Image is a Half-plane [R03]

A half-plane or

ε-far from a half-plane?

O(1/ε) time



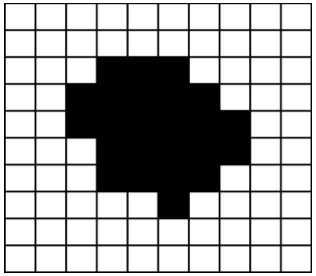
Other Results on Properties of Images

- Pixel Model

  Convexity [Berman Murzabulatov R]

  Convex or ε-far from convex?

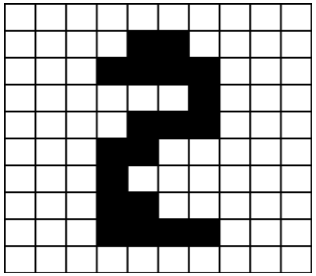
  $O(1/\varepsilon)$ time



  Connectedness [Berman Murzabulatov R]

  Connected or ε-far from connected?

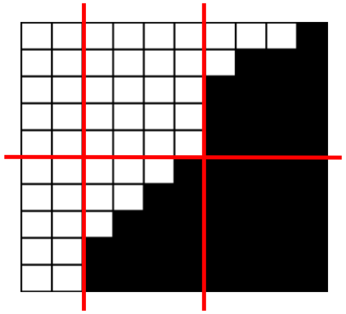
  $O(1/\varepsilon^{3/2}\sqrt{\log 1/\varepsilon})$ time



  Partitioning [Kleiner Keren Newman 10]

  Can be partitioned according to a template or is ε-far?

  time independent of image size



- Properties of sparse images [Ron Tsur 10]

Testing if a List is Sorted

Input: a list of n numbers $x_1, x_2, \ldots, x_n$

- Question: Is the list sorted?

  Requires reading entire list: $\Omega(n)$ time

- Approximate version: Is the list sorted or ϵ-far from sorted?

  (An ϵ fraction of x_i's have to be changed to make it sorted.)

  [Ergün Kannan Kumar Rubinfeld Viswanathan 98, Fischer 01]: $O((\log n)/\epsilon)$ time

  $\Omega(\log n)$ queries

- Attempts:

  1. Test: Pick a random i and reject if $x_i > x_{i+1}$.

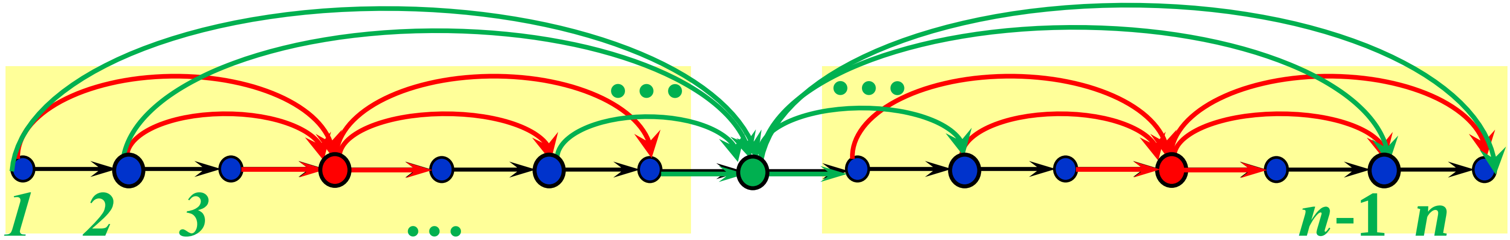
     Fails on: 1 1 1 1 1 1 1 0 0 0 0 0 0 0 $\leftarrow$ 1/2-far from sorted

  2. Test: Pick random $i < j$ and reject if $x_i > x_j$.

     Fails on: 1 0 2 1 3 2 4 3 5 4 6 5 7 6 $\leftarrow$ 1/2-far from sorted

Is a list sorted or ϵ-far from sorted?

Idea: Associate positions in the list with vertices of the directed line.



Construct a graph (2-spanner)

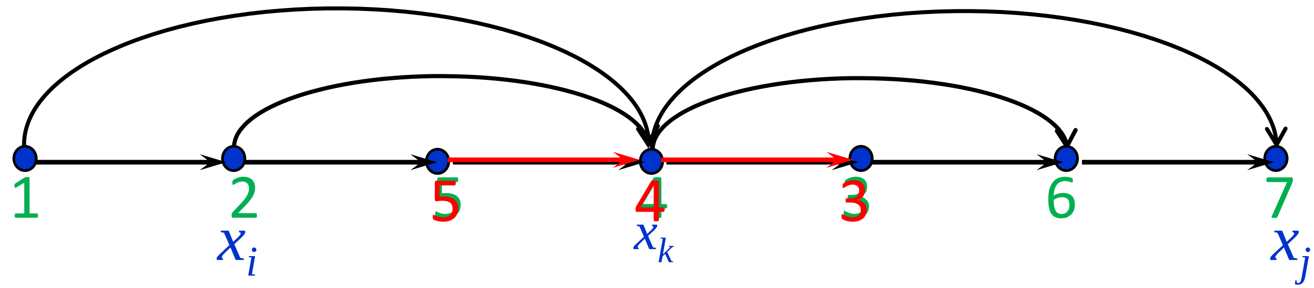
≤ $n \log n$ edges

- by adding a few "shortcut" edges (i, j) for $i < j$
- where each pair of vertices is connected by a path of length at most 2

Is a list sorted or ϵ-far from sorted?

Test [Dodis Goldreich Lehman Raskhodnikova Ron Samorodnitsky 99]

Pick a random edge (x_i,x_j) from the 2-spanner and **reject** if $x_i > x_j$.



Analysis:

- Call an edge (x_i,x_j) **violated** if $x_i > x_j$, and **good** otherwise.
- If x_i is an endpoint of a **violated** edge, call it **bad**. Otherwise, call it **good**.

Claim 1. All **good** numbers x_i are sorted.

Proof: Consider any two good numbers, x_i and x_j.

They are connected by a path of (at most) two **good** edges (x_i,x_k), (x_k,x_j).

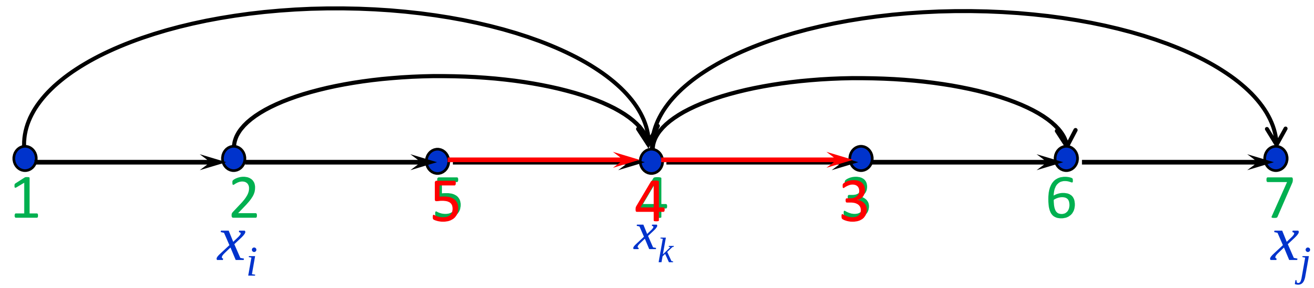
$\Rightarrow$ $x_i \leq x_k$ and $x_k \leq x_j$

$\Rightarrow$ $x_i \leq x_j$

Is a list sorted or ϵ-far from sorted?

Test [Dodis Goldreich Lehman Raskhodnikova Ron Samorodnitsky 99]

Pick a random edge (x_i, x_j) from the 2-spanner and **reject** if $x_i > x_j$.



Analysis:

- Call an edge (x_i, x_j) **violated** if $x_i > x_j$, and **good** otherwise.
- If x_i is an endpoint of a **bad** edge, call it **bad**. Otherwise, call it **good**.

Claim 1. All **good** numbers x_i are sorted.

Claim 2. An ϵ-far list **violates** $\geq \epsilon/(2 \log n)$ fraction of edges in 2-spanner.

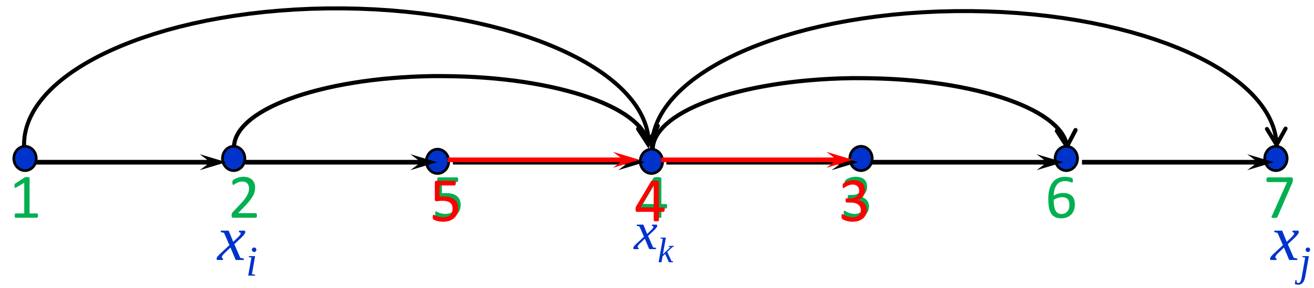
Proof: If a list is ϵ-far from sorted, it has $\geq \epsilon n$ **bad** numbers. (Claim 1)

- Each **violated** edge contributes 2 **bad** numbers.
- 2-spanner has $\geq \epsilon n/2$ **violated** edges out of $\leq n \log n$.

Is a list sorted or ϵ-far from sorted?

Test [Dodis Goldreich Lehman Raskhodnikova Ron Samorodnitsky 99]

Pick a random edge (x_i,x_j) from the 2-spanner and **reject** if $x_i > x_j$.



Analysis:

- Call an edge (x_i,x_j) **violated** if $x_i > x_j$, and **good** otherwise.

Claim 2. An ϵ-far list **violates** $\geq \epsilon/(2 \log n)$ fraction of edges in 2-spanner.

By Witness Lemma, it suffices to sample $(4 \log n)/\epsilon$ edges from 2-spanner.

Algorithm

Sample $(4 \log n)/\epsilon$ edges (x_i,x_j) from the 2-spanner and **reject** if $x_i > x_j$.

Guarantee: All sorted lists are accepted.

All lists that are ϵ-far from sorted are rejected with probability $\geq 2/3$.

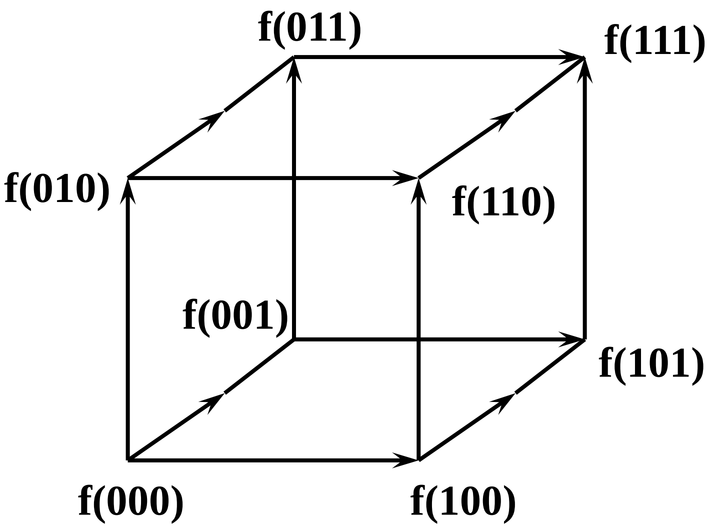
Time: $O((\log n)/\epsilon)$

Basic Properties of Functions

Boolean Functions $f : \{0,1\}^n \to \{0,1\}$

Graph representation:

n-dimensional hypercube



- vertices: bit strings of length n
- edges: (x, y) is an edge if y can be obtained from x by increasing one bit from 0 to 1

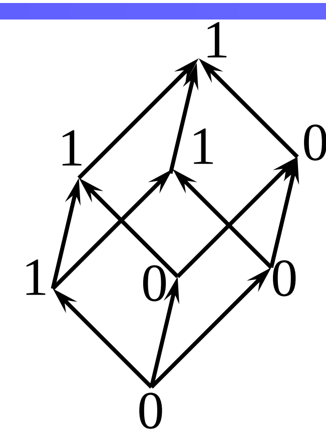
x	001001
y	011001

- each vertex x is labeled with $f(x)$

Monotonicity of Functions

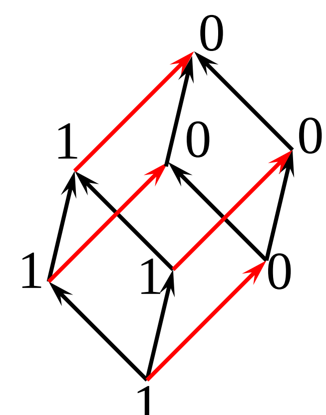
[Goldreich Goldwasser Lehman Ron Samorodnitsky,
Dodis Goldreich Lehman Raskhodnikova Ron Samorodnitsky
Fischer Lehman Newman Raskhodnikova Rubinfeld Samorodnitsky]

- A function $f : \{0,1\}^n \to \{0,1\}$ is monotone if increasing a bit of x does not decrease $f(x)$.



monotone

- Is f monotone or ε-far from monotone (f has to change on many points to become monontone)?
  - Edge $x \to y$ is violated by f if $f(x) > f(y)$.



$\frac{1}{2}$-far from monotone

Time:

- $O(n/\varepsilon)$, logarithmic in the size of the input, 2^n
- $\Omega(\sqrt{n}/\varepsilon)$ for restricted class of tests
- Recent: $\Theta(\sqrt{n}/\varepsilon^2)$ for nonadaptive tests

[Khot Minzer Safra 15, Chen De Servidio Tang 15]

Monotonicity Test [GGLRS, DGLRRS]

Idea: Show that functions that are far from monotone violate many edges.

EdgeTest (f, ε)

1. Pick $2n/\varepsilon$ edges (x, y) uniformly at random from the hypercube.
2. **Reject** if some (x, y) is violated (i.e. $f(x) > f(y)$). Otherwise, **accept**.

Analysis

- If f is monotone, EdgeTest always accepts.
- If f is ε-far from monotone, by Witness Lemma, it suffices to show that $\geq \varepsilon/n$ fraction of edges (i.e., $\frac{\varepsilon}{n} \cdot 2^{n-1} n = \varepsilon 2^{n-1}$ edges) are violated by f.
  - Let $V(f)$ denote the number of edges violated by f.

Contrapositive: If $V(f) < \varepsilon\, 2^{n-1}$,

f can be made monotone by changing $< \varepsilon\, 2^n$ values.

Repair Lemma

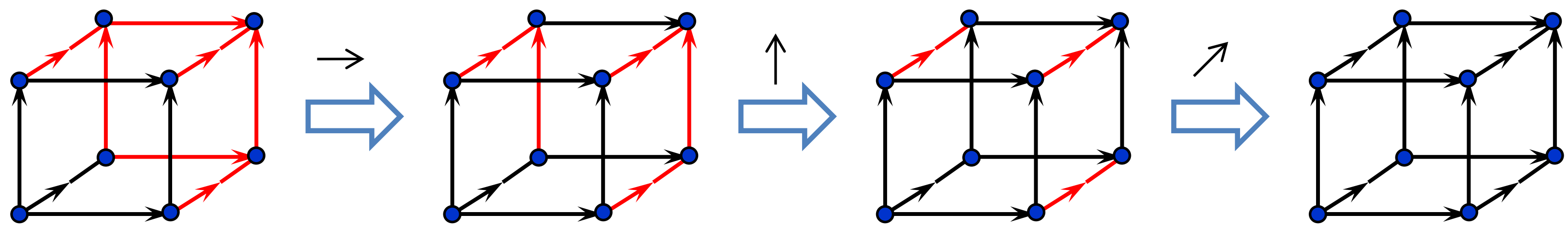
f can be made monotone by changing $\leq 2 \cdot V(f)$ values.

Repair Lemma: Proof Idea

Repair Lemma

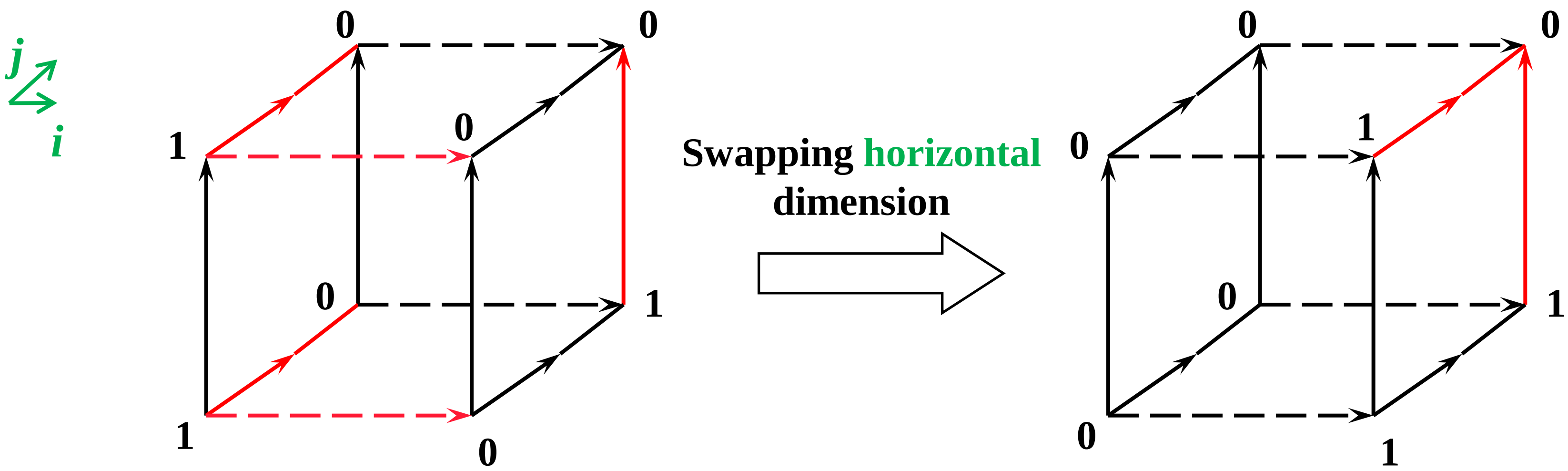
f can be made monotone by changing $\leq 2 \cdot V(f)$ values.

Proof idea: Transform f into a monotone function by repairing edges in one dimension at a time.



Repairing Violated Edges in One Dimension

Swap violated edges $1 \to 0$ in one dimension to $0 \to 1$.



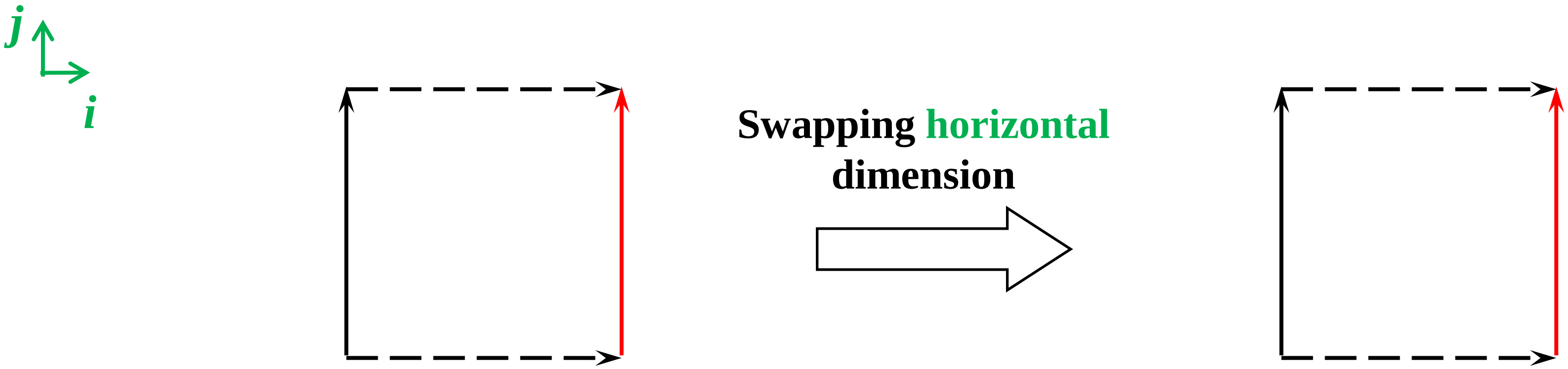
Let V_j = # of violated edges in dimension j

Claim. Swapping in dimension i does not increase V_j for all dimensions $j \neq i$

Enough to prove the claim for squares

Proof of The Claim for Squares

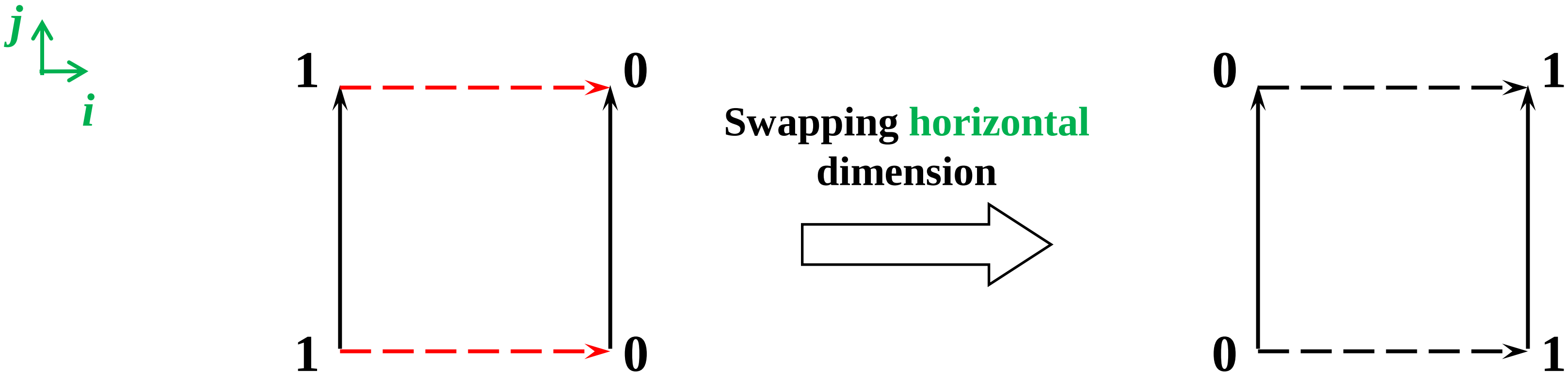
Claim. Swapping in dimension i does not increase V_j for all dimensions $j \neq i$



- If no horizontal edges are violated, no action is taken.

Proof of The Claim for Squares

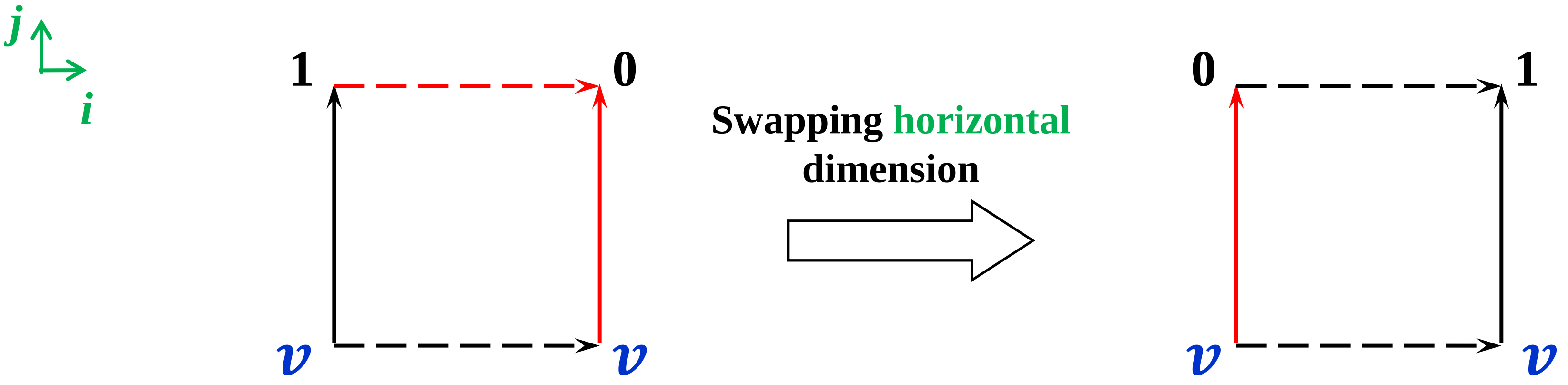
Claim. Swapping in dimension i does not increase V_j for all dimensions $j \neq i$



- If both horizontal edges are violated, both are swapped, so the number of vertical violated edges does not change.

Proof of The Claim for Squares

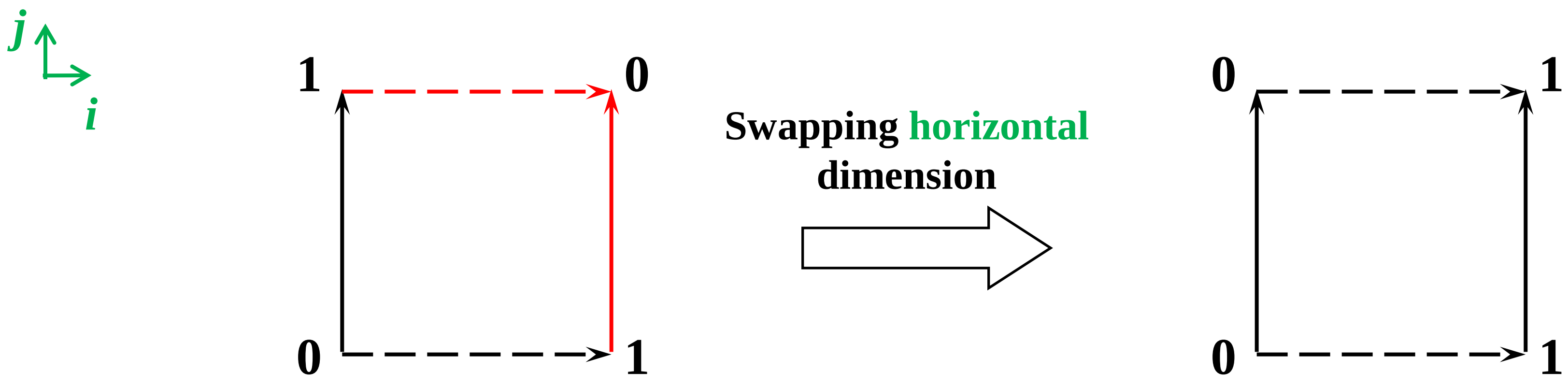
Claim. Swapping in dimension i does not increase V_j for all dimensions $j \neq i$



- Suppose one (say, top) horizontal edge is violated.
- If both bottom vertices have the same label, the vertical edges get swapped.

Proof of The Claim for Squares

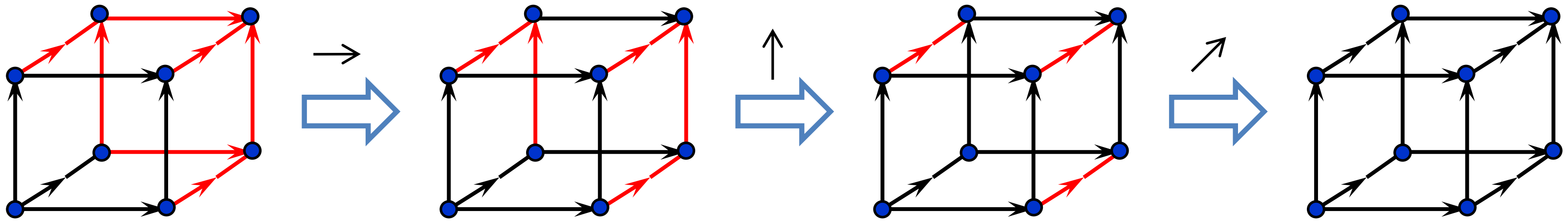
Claim. Swapping in dimension i does not increase V_j for all dimensions $j \neq i$ ✔



- Suppose one (say, top) horizontal edge is violated.
- If both bottom vertices have the same label, the vertical edges get swapped.
- Otherwise, the bottom vertices are labeled 0→1, and the vertical violation is repaired.

Proof of The Claim for Squares

Claim. Swapping in dimension i does not increase V_j for all dimensions $j \neq i$



After we perform swaps in all dimensions:

- f becomes monotone
- # of values changed:

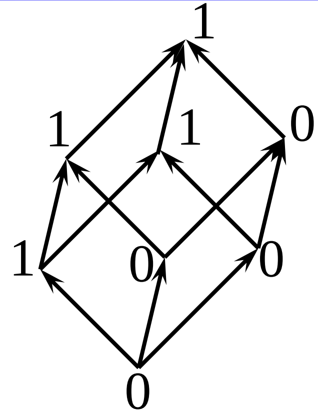
  $2 \cdot V_1 + 2 \cdot (\text{\# violated edges in dim 2 after swapping dim 1})$
  $+ 2 \cdot (\text{\# violated edges in dim 3 after swapping dim 1 and 2})$
  $+ \ldots \leq 2 \cdot V_1 + 2 \cdot V_2 + \cdots 2 \cdot V_n = 2 \cdot V(f)$

Repair Lemma

f can be made monotone by changing $\leq 2 \cdot V(f)$ values.

Improve the bound by a factor of 2.

Testing if a Functions $f : \{0,1\}^n \to \{0,1\}$ is monotone



monotone

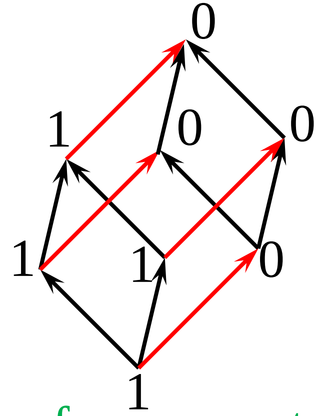
Monotone or

ε-far from monotone?

O(n/ε) time ✓

(logarithmic in the size

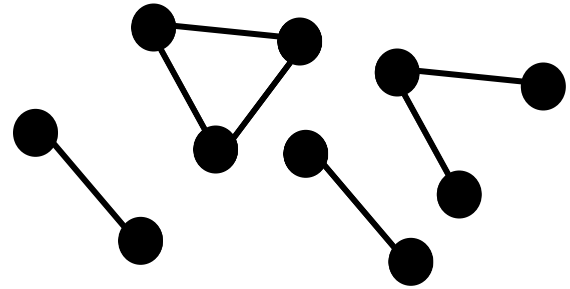
of the input)



$\frac{1}{2}$-far from monotone

Graph Properties

Testing if a Graph is Connected [Goldreich Ron]



Input: a graph $G = (V, E)$ on n vertices

- in adjacency lists representation
  (a list of neighbors for each vertex)
- maximum degree *d*, i.e., adjacency lists of length *d* with some empty entries

Query (v, i), where $v \in V$ and $i \in [d]$: entry i of adjacency list of vertex v

Exact Answer: Ω(dn) time

- Approximate version:

Is the graph connected or ϵ-far from connected?

$$\text{dist}(G_1, G_2) = \frac{\#\ of\ entires\ in\ adjacency\ lists\ on\ which\ G_1\ and\ G_2\ differ}{dn}$$

Time: $O\left(\frac{1}{\varepsilon^2 d}\right)$ today

No dependence on n!

Testing Connectedness: Algorithm

Connectedness Tester(G, d, ε)

1. **Repeat** s=8/εd times:
2. pick a random vertex u
3. determine if connected component of u is small:

   perform BFS from u, stopping after at most 4/εd new nodes
4. **Reject** if a small connected component was found, otherwise **accept.**

Run time: O($d/\varepsilon^2 d^2$)=O($1/\varepsilon^2 d$)

Analysis:

- Connected graphs are always accepted.
- Remains to show:

  If a graph is ϵ-far from connected, it is rejected with probability $\geq \frac{2}{3}$

Testing Connectedness: Analysis

Claim 1

If G is ε-far from connected, it has $\geq \frac{\varepsilon d n}{2}$ connected components.

Claim 2

If G is ε-far from connected, it has $\geq \frac{\varepsilon d n}{4}$ connected components of size at most 4/εd.

- If Claim 2 holds, at least $\frac{\varepsilon d n}{4}$ nodes are in small connected components.
- By Witness lemma, it suffices to sample $\frac{2 \cdot 4}{\varepsilon d n / n} = \frac{8}{\varepsilon d}$ nodes to detect one from a small connected component.

Testing Connectedness: Proof of Claim 1

Claim 1

If G is ε-far from connected, it has $\geq \frac{\varepsilon dn}{2}$ connected components.

We prove the contrapositive:

If G has < $\frac{\varepsilon dn}{2}$ connected components, one can make G connected by modifying < ε fraction of its representation, i.e., < εdn entries.

- If there are no degree restrictions, k components can be connected by adding k-1 edges, each affecting 2 nodes. Here, $k < \frac{\varepsilon dn}{2}$, so $2k-2 < \varepsilon dn$.
- What if adjacency lists of all vertices in a component are full,

  i.e., all vertex degrees are d?

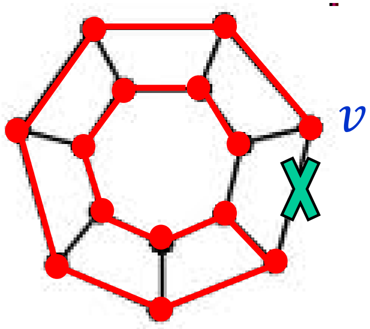
Freeing up an Adjacency List Entry

Claim 1

If G is ε-far from connected, it has $\geq \frac{\varepsilon d n}{2}$ connected components.

What if adjacency lists of all vertices in a component are full,

i.e., all vertex degrees are d?



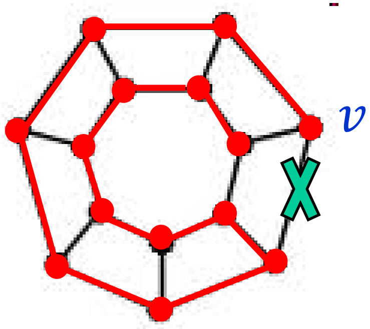
- Consider an MST of this component.
- Let v be a leaf of the MST.
- Disconnect v from a node other than its parent in the MST.
- Two entries are changed while keeping the same number of components.

Freeing up an Adjacency List Entry

Claim 1

If G is ε-far from connected, it has $\geq \frac{\varepsilon d n}{2}$ connected components.

What if adjacency lists of all vertices in a component are full,
i.e., all vertex degrees are d?



- Apply this to each component that <2 free spots in adjacency lists.
- Now we can connect all the components using the freed up spots while ensuring that we never change more than 2 spots per component.
- Thus, k components can be connected by changing 2k spots.

Here, $k < \frac{\varepsilon d n}{4}$, so $2k < \varepsilon d n$.

Testing Connectedness: Proof of Claim 2

Claim 1

If G is ε-far from connected, it has $\geq \frac{\varepsilon dn}{2}$ connected components.

Claim 2

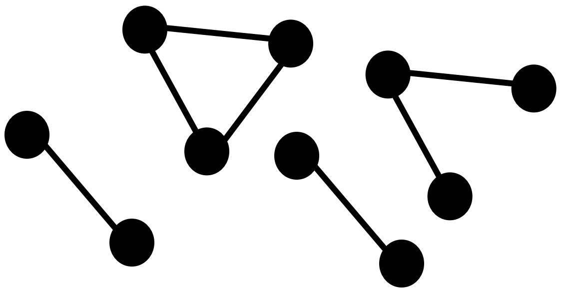
If G is ε-far from connected, it has $\geq \frac{\varepsilon dn}{4}$ connected components of size at most 4/εd.

- If Claim 1 holds, there are at least $\frac{\varepsilon dn}{2}$ connected components.
- Their average size $\leq \frac{n}{\varepsilon dn/2} = \frac{2}{\varepsilon d}$.
- By an averaging argument (or Markov inequality), at least half of the components are of size at most twice the average.

Testing if a Graph is Connected [Goldreich Ron]

Input: a graph $G = (V, E)$ on n vertices

- in adjacency lists representation
  (a list of neighbors for each vertex)
- maximum degree d



Connected or

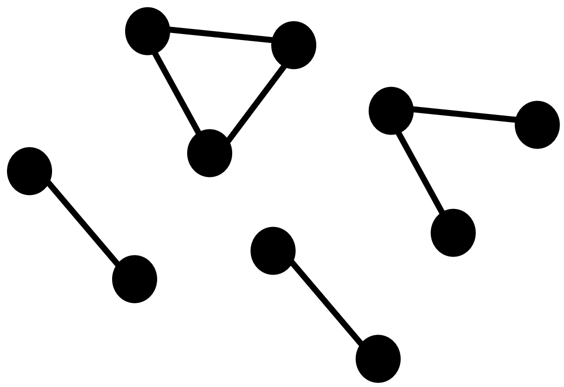
ε-far from connected?

$O\left(\frac{1}{\varepsilon^2 d}\right)$ time ✓

(no dependence on n)

Approximating # of Connected Components

[Chazelle Rubinfeld Trevisan]



Input: a graph $G = (V, E)$ on n vertices

- in adjacency lists representation
  (a list of neighbors for each vertex)
- maximum degree *d*

Exact Answer: Ω(dn) time

Additive approximation: # of CC ±εn

with probability $\geq$ 2/3

Time:

- Known: $O\left(\frac{d}{\varepsilon^2}\log\frac{1}{\varepsilon}\right)$, $\Omega\left(\frac{d}{\varepsilon^2}\right)$
- Today: $O\left(\frac{d}{\varepsilon^3}\right)$.

Partially based on slides by Ronitt Rubinfeld:
http://stellar.mit.edu/S/course/6/fa10/6.896/courseMaterial/topics/topic3/lectureNotes/lecst11/lecst11.pdf

Approximating # of CCs: Main Idea

- Let C = number of components
- For every vertex u, define
  n_u = number of nodes in *u's component*
  - for each component A: $\sum_{u \in A} \frac{1}{n_u} = 1$

$$\sum_{u \in V} \frac{1}{n_u} = C$$

Breaks C up into contributions of different nodes

- Estimate this sum by estimating n_u's for a few random nodes
  - If u's component is small, its size can be computed by BFS.
  - If u's component is big, then $1/n_u$ is small, so it does not contribute much to the sum
  - Can stop BFS after a few steps

Similar to property tester for connectedness [Goldreich Ron]

Approximating # of CCs: Algorithm

Estimating n_u = the number of nodes in u's component:

- Let estimate $\hat{n}_u = \min\left\{n_u, \frac{2}{\varepsilon}\right\}$
  - When u's component has $\leq 2/\varepsilon$ nodes, $\hat{n}_u = n_u$
  - Else $\hat{n}_u = 2/\varepsilon$, and so $0 < \frac{1}{\hat{n}_u} - \frac{1}{n_u} < \frac{1}{\hat{n}_u} = \frac{\varepsilon}{2}$

$$\left|\frac{1}{\hat{n}_u} - \frac{1}{n_u}\right| \leq \frac{\varepsilon}{2}$$

- Corresponding estimate for C is $\hat{C} = \sum_{u \in V} \frac{1}{\hat{n}_u}$. It is a good estimate:

$$\left|\hat{C} - C\right| = \left|\sum_{u \in V} \frac{1}{\hat{n}_u} - \sum_{u \in V} \frac{1}{n_u}\right| \leq \sum_{u \in V} \left|\frac{1}{\hat{n}_u} - \frac{1}{n_u}\right| \leq \frac{\varepsilon n}{2}$$

APPROX_#_CCs (G, d, ε)

1. **Repeat** s=Θ(1/ε^2) times:
2. pick a random vertex u
3. compute $\hat{n}_u$ via BFS from u, stopping after at most 2/ε new nodes
4. **Return** $\tilde{C}$ = (average of the values $1/\hat{n}_u$) $\cdot\, n$

Run time: O(d /ε^3)

Approximating # of CCs: Analysis

Want to show: $\Pr\left[\left|\tilde{C} - \hat{C}\right| > \frac{\varepsilon n}{2}\right] \leq \frac{1}{3}$ ✓

Hoeffding Bound

Let $Y_1, \dots, Y_s$ be independently distributed random variables in [0,1] and let $Y = \sum_{i=1}^{s} Y_i$ (sample sum). Then $\Pr[|Y - E[Y]| \geq \delta] \leq 2e^{-2\delta^2/s}$.

Let $Y_i = 1/\hat{n}_u$ for the i^{th} vertex u in the sample

- $Y = \sum_{i=1}^{s} Y_i = \frac{s\tilde{C}}{n}$ and $E[Y] = \sum_{i=1}^{s} E[Y_i] = s \cdot E[Y_1] = s \cdot \frac{1}{n} \sum_{u \in V} \frac{1}{\hat{n}_u} = \frac{s\hat{C}}{n}$

$$\Pr\left[\left|\tilde{C} - \hat{C}\right| > \frac{\varepsilon n}{2}\right] = \Pr\left[\left|\frac{n}{s} Y - \frac{n}{s} E[Y]\right| > \frac{\varepsilon n}{2}\right] = \Pr\left[\left|Y - E[Y]\right| > \frac{\varepsilon s}{2}\right] \leq 2e^{-\frac{\varepsilon^2 s}{2}}$$

- Need $s = \Theta\left(\frac{1}{\varepsilon^2}\right)$ samples to get probability $\leq \frac{1}{3}$

Approximating # of CCs: Analysis

So far: $$\left|\hat{C} - C\right| \leq \frac{\varepsilon n}{2}$$

$$\Pr\left[\left|\tilde{C} - \hat{C}\right| > \frac{\varepsilon n}{2}\right] \leq \frac{1}{3}$$

- With probability $\geq \frac{2}{3}$,

$$\left|\tilde{C} - C\right| \leq \left|\tilde{C} - \hat{C}\right| + \left|\hat{C} - C\right| \leq \frac{\varepsilon n}{2} + \frac{\varepsilon n}{2} \leq \varepsilon n$$

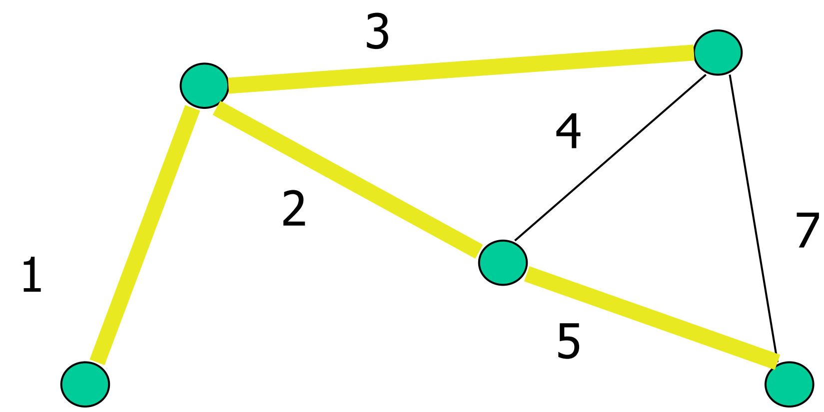
Summary:

The number of connected components in n-vetex graphs of degree at most d can be estimated within $\pm\varepsilon n$ in time $O\left(\frac{d}{\varepsilon^3}\right)$.

Minimum spanning tree (MST)

- What is the cheapest way to connect all the dots?

Input: a weighted graph
with n vertices and m edges



- Exact computation:
  - Deterministic $O(m \cdot \text{inverse-Ackermann}(m))$ time [Chazelle]
  - Randomized $O(m)$ time [Karger Klein Tarjan]

Partially based on slides by Ronitt Rubinfeld:
http://stellar.mit.edu/S/course/6/fa10/6.896/courseMaterial/topics/topic3/lectureNotes/lecst11/lecst11.pdf

Approximating MST Weight in Sublinear Time

[Chazelle Rubinfeld Trevisan]

Input: a graph $G = (V, E)$ on n vertices

- in adjacency lists representation
- maximum degree *d* and maximum allowed weight *w*
- weights in {1,2,…,*w*}

Output: (1+ ε)-approximation to MST weight, w_{MST}

Time:

- Known: $O\left(\frac{dw}{\varepsilon^3}\log\frac{dw}{\varepsilon}\right), \Omega\left(\frac{dw}{\varepsilon^2}\right)$
- Today: $O\left(\frac{dw^4 \log w}{\varepsilon^3}\right)$

No dependence on n!

Idea Behind Algorithm

- Characterize MST weight in terms of number of connected components in certain subgraphs of *G*

- Already know that number of connected components can be estimated quickly

MST and Connected Components: Warm-up

- Recall Kruskal's algorithm for computing MST exactly.

Suppose all weights are 1 or 2. Then MST weight

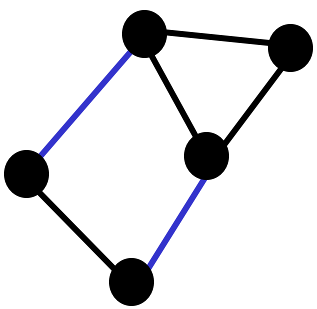
= (# weight-1 edges in MST) + 2 · (# weight-2 edges in MST)

$= \; n - 1 \; +$ (# of weight-2 edges in MST)

MST has $n - 1$ edges

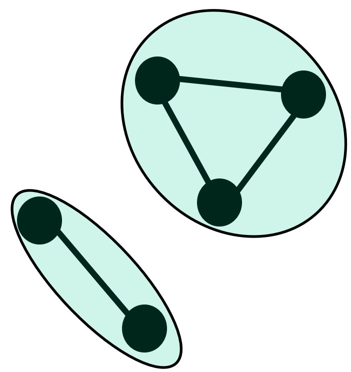
$= \; n - 1 \; +$ (# of CCs induced by weight-1 edges) -1

By Kruskal

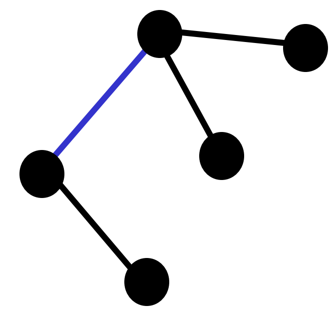


weight 1

weight 2



connected components
induced by weight-1 edges



MST

MST and Connected Components

In general: Let G_i = subgraph of G containing all edges of weight $\leq i$

C_i = number of connected components in G_i

Then MST has $C_i - 1$ edges of weight $> i$.

Claim

$$w_{MST}(G) = n - w + \sum_{i=1}^{w-1} C_i$$

✔

- Let β_i be the number of edges of weight $> i$ in MST
- Each MST edge contributes 1 to w_{MST}, each MST edge of weight >1 contributes 1 more, each MST edge of weight >2 contributes one more, …

$$w_{MST}(G) = \sum_{i=0}^{w-1} \beta_i = \sum_{i=0}^{w-1} (C_i - 1) = -w + \sum_{i=0}^{w-1} C_i = n - w + \sum_{i=1}^{w-1} C_i$$

Algorithm for Approximating w_{MST}

APPROX_MSTweight (G, w, d, ε)

1. **For** $i = 1$ to $w - 1$ **do**:
2. $\tilde{C}_i \leftarrow$APPROX_#CCs$(G_i, d, \varepsilon/\text{w})$.
3. **Return** $\widetilde{w}_{MST} = n - w + \sum_{i=1}^{w-1} \tilde{C}_i$.

Claim. $w_{MST}(G) = n - w + \sum_{i=1}^{w-1} C_i$

Analysis:

- Suppose all estimates of C_i's are good: $\left|\tilde{C}_i - C_i\right| \le \frac{\varepsilon}{w}\, n$.

  Then $|\widetilde{w}_{MST} - w_{MST}| = |\sum_{i=1}^{w-1}(\tilde{C}_i - C_i)| \le \sum_{i=1}^{w-1} |\tilde{C}_i - C_i| \le w \cdot \frac{\varepsilon}{w}\, n = \varepsilon n$
- Pr[all $w - 1$ estimates are good]$\ge (2/3)^{w-1}$
- Not good enough! Need error probability $\le \frac{1}{3w}$ for each iteration
- Then, by Union Bound, Pr[error]$\le w \cdot \frac{1}{3w} = \frac{1}{3}$

Can amplify success probability of any algorithm by repeating it and taking the median answer.

Can take more samples in APPROX_#CCs. What's the resulting run time?

Multiplicative Approximation for w_{MST}

For MST cost, additive approximation $\Longrightarrow$ multiplicative approximation

$$w_{MST} \geq n-1 \quad \Longrightarrow \quad w_{MST} \geq n/2 \text{ for } n \geq 2$$

- εn-additive approximation:

$$w_{MST} - \varepsilon n \leq \widehat{w}_{MST} \leq w_{MST} + \varepsilon n$$

- $(1 \pm 2\varepsilon)$-multiplicative approximation:

$$w_{MST}(1-2\varepsilon) \leq w_{MST} - \varepsilon n \leq \widehat{w}_{MST} \leq w_{MST} + \varepsilon n \leq w_{MST}(1+2\varepsilon)$$