CIS 700:
"algorithms for Big Data"

Lecture 10:
Massively Parallel Algorithms

Slides at http://grigory.us/big-data-class.html

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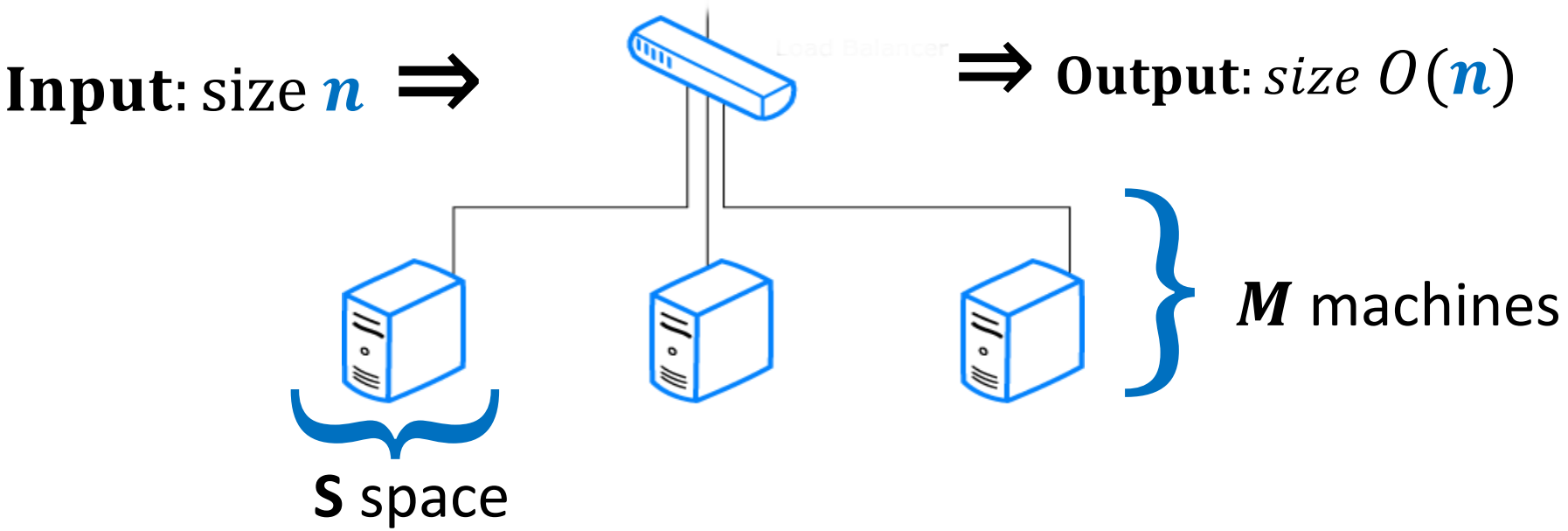
Computational Model

- **Input**: size **n**
- $\boldsymbol{M}$ machines, space $\boldsymbol{S}$ on each ($\boldsymbol{S} = \boldsymbol{n}^{1-\epsilon}$, $0 < \epsilon < 1$)
  - Constant overhead in total space: $\boldsymbol{M} \cdot \boldsymbol{S} = O(\boldsymbol{n})$
- **Output**: solution to a problem (often size O($\boldsymbol{n}$))
  - Doesn't fit on a single machine ($\boldsymbol{S} \ll \boldsymbol{n}$)

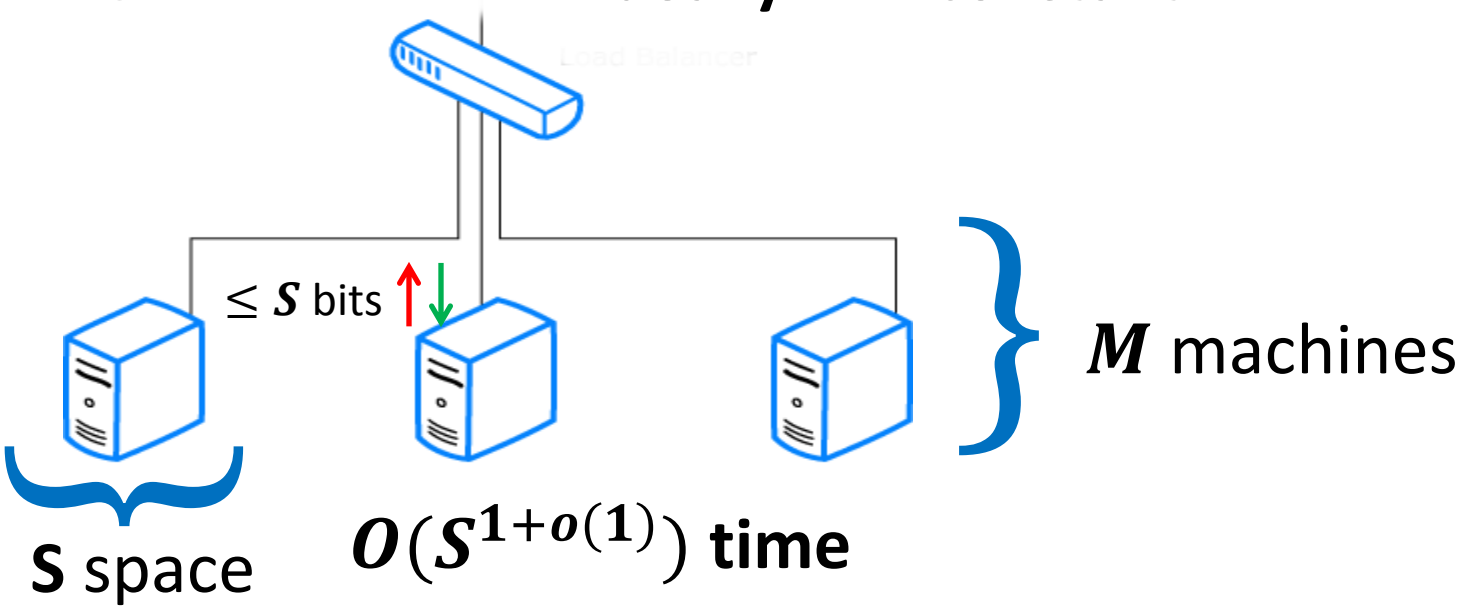


Computational Model

- Computation/Communication in $\boldsymbol{R}$ rounds:
  - Every machine performs a **near-linear time** computation => Total running time $O(\boldsymbol{n^{1+o(1)}R})$
  - Every machine **sends/receives at most $\boldsymbol{S}$ bits** of information => Total communication $O(\boldsymbol{nR})$.

Goal: Minimize $\boldsymbol{R}$.

Ideally: $\boldsymbol{R}$ = constant.

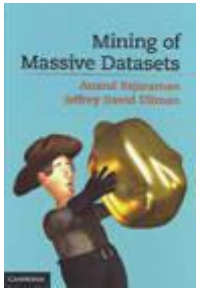


MapReduce-style computations

YAHOO! Google

What I won't discuss today

- PRAMs (**shared memory**, multiple processors) (see e.g. [Karloff, Suri, Vassilvitskii'10])
  - Computing XOR requires $\widetilde{\Omega}(\log n)$ rounds in CRCW PRAM
  - Can be done in $O(\log_s n)$ rounds of MapReduce
- Pregel-style systems, Distributed Hash Tables (see e.g. Ashish Goel's class notes and papers)
- Lower-level implementation details (see e.g. Rajaraman-Leskovec-Ullman book)



Models of parallel computation

- **Bulk-Synchronous Parallel Model** (BSP) **[Valiant,90]**

  Pro: Most general, generalizes all other models

  Con: Many parameters, hard to design algorithms

- **Massive Parallel Computation** [Feldman-Muthukrishnan-Sidiropoulos-Stein-Svitkina'07, Karloff-Suri-Vassilvitskii'10, **Goodrich-Sitchinava-Zhang'11, ..., Beame, Koutris, Suciu'13**]

  Pros:

  - Inspired by **modern** systems (Hadoop, MapReduce, Dryad, …)
  - Few parameters, **simple** to design algorithms
  - **New algorithmic ideas**, robust to the exact model specification
  - **# Rounds** is an information-theoretic measure => can prove unconditional lower bounds
  - Between **linear sketching** and **streaming with sorting**

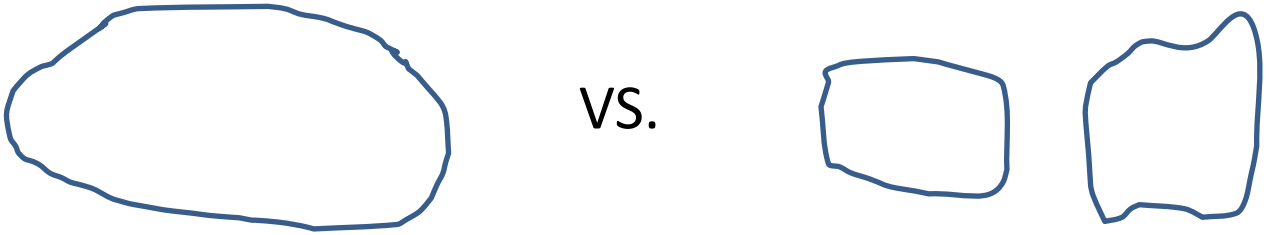
Sorting: Terasort

- Sorting $\boldsymbol{n}$ keys on $\boldsymbol{M = O(n^{1-\alpha})}$ machines
  - Would like to partition keys uniformly into blocks: first $\boldsymbol{n/M}$, second $\boldsymbol{n/M}$, etc.
  - Sort the keys locally on each machine
- Build an approximate histogram:
  - Each machine takes a sample of size $\boldsymbol{s}$
  - All $\boldsymbol{M * s \leq S = n^{\alpha}}$ samples are sorted locally
  - Blocks are computed based on the samples
- By Chernoff bound $\mathbf{M} * \boldsymbol{s} = O\left(\frac{log\, \boldsymbol{n}}{\boldsymbol{\epsilon^2}}\right)$ samples suffice to compute al block sizes with $\pm\epsilon\boldsymbol{n}$ error
- Take $\epsilon = \frac{\boldsymbol{n^{\alpha-1}}}{2}$: error $\mathrm{O}(\boldsymbol{S})$; $\mathbf{M} * \boldsymbol{s} = \widetilde{O}(\boldsymbol{n^{2-2\alpha}}) = \boldsymbol{O(M^2)} \leq \boldsymbol{O(n^{\alpha})}$ for $\alpha \geq 2/3$

Algorithms for Graphs

- **Dense graphs** vs. sparse graphs
  - **Dense**: $S \gg |V|$
    - Linear sketching: one round
    - "Filtering" (Output fits on a single machine) [Karloff, Suri Vassilvitskii, SODA'10; Ene, Im, Moseley, KDD'11; Lattanzi, Moseley, Suri, Vassilvitskii, SPAA'11; Suri, Vassilvitskii, WWW'11]
  - Sparse: $S \ll |V|$ (or $S \ll$ solution size)

    Sparse graph problems appear hard (**Big open question**: connectivity in $o(\log n)$ rounds?)



Algorithm for Connectivity

- Version of Boruvka's algorithm
- Repeat $O(\log n)$ times:
  - Each component chooses a neighboring component
  - All pairs of chosen components get merged
- How to avoid **chaining**?
- If the graph of components is bipartite and only one side gets to choose then no chaining
- **Randomly** assign components to the sides

Algorithm for Connectivity: Setup

Data: **N** edges of an undirected graph.

Notation:

- For $v \in V$ let $\pi(v)$ be its id in the data
- $\Gamma(S) \equiv$ set of neighbors of a subset of vertices S⊆V.

Labels:

- Algorithms assigns a label $\ell(v)$ to each v.
- Let $L_v \subseteq V$ be the set of vertices with the label $\ell(v)$ (invariant: subset of the connected component containing v).

Active vertices:

- Some vertices will be called **active**.
- Every set L_v will have exactly one active vertex.

Algorithm for Connectivity

- Mark every vertex as **active** and let $\ell(v) = \pi(v)$.
- For phases $i = 1,2, \ldots, O(\log N)$ do:
  - Call each **active** vertex a **leader** with probability 1/2. If v is a **leader**, mark all vertices in L_v as **leaders**.
  - For every **active** **non-leader** vertex w, find the smallest **leader**(with respect to π) vertex $w^\star \in \Gamma(L_w)$.
  - If $w^\star$ is not empty, mark w **passive** and relabel each vertex with label w by $w^\star$.
- Output the set of CCs, where vertices having the same label according to ℓ are in the same component.

Algorithm for Connectivity: Analysis

- If $\ell(u) = \ell(v)$ then u and v are in the same CC.
- Unique labels w.h.p after $O(\log N)$ phases.
- For every CC # active vertices reduces by a constant factor in every phase.
  - Half of the active vertices declared as non-leaders.
  - Fix an active **non-leader** vertex $\boldsymbol{v}$.
  - If at least two different labels in the CC of v then there is an edge $(\boldsymbol{v'}, \boldsymbol{u})$ such that $\ell(\boldsymbol{v}) = \ell(\boldsymbol{v'})$ and $\ell(\boldsymbol{v'}) \neq \ell(\boldsymbol{u})$.
  - $\boldsymbol{u}$ marked as a **leader** with probability 1/2; in expectation half of the active non-leader vertices will change their label.
  - Overall, expect 1/4 of labels to disappear.
  - By Chernoff after $O(\log N)$ phases # of active labels in every connected component will drop to one w.h.p.

Algorithm for Connectivity: Implementation Details

- Distributed data structure of size $O(|V|)$ to maintain labels, ids, leader/non-leader status, etc.
  - O(1) rounds per stage to update the data structure
- Edges stored locally with all auxiliary info
  - Between stages: use distributed data structure to update local info on edges
- For every **active non-leader** vertex w, find the smallest **leader** (w.r.t π) vertex $w^\star \in \Gamma(L_w)$
  - Each (**non-leader, leader**) edges sends an update to the distributed data structure
- Much faster with Distributed Hash Table Service (DHT) [Kiveris, Lattanzi, Mirrokni, Rastogi, Vassilvitskii'14]

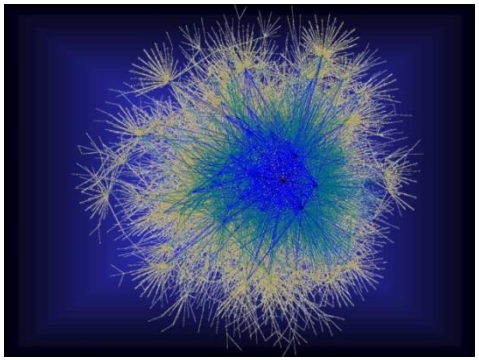
Applications

- Using same reductions as in streaming:
  - Bipartiteness
  - k-connectivity
  - Cut-sparsification

Approximating Geometric Problems in Parallel Models

Geometric graph (implicit):

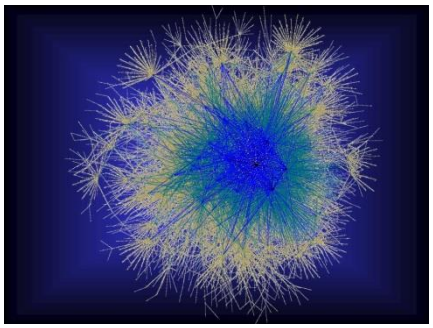
Euclidean distances between **n** points in $\mathbb{R}^{d}$



Already have solutions for old NP-hard problems (Traveling Salesman, Steiner Tree, etc.)

- Minimum Spanning Tree (clustering, vision)
- Minimum Cost Bichromatic Matching (vision)

Geometric Graph Problems

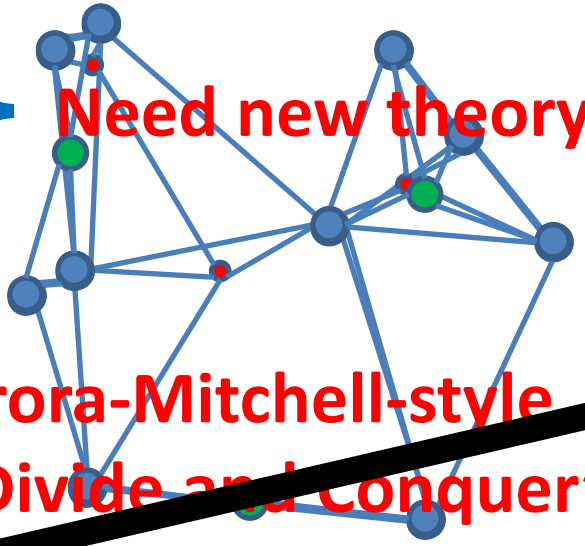


Combinatorial problems on graphs in $\mathbb{R}^d$

Polynomial time ("easy")

- Minimum Spanning Tree
- Earth-Mover Distance =

Min Weight Bi-chromatic Matching



Need new theory!

NP-hard ("hard")

- Steiner Tree
- Traveling Salesman
- Clustering (k-medians, facility location, etc.)

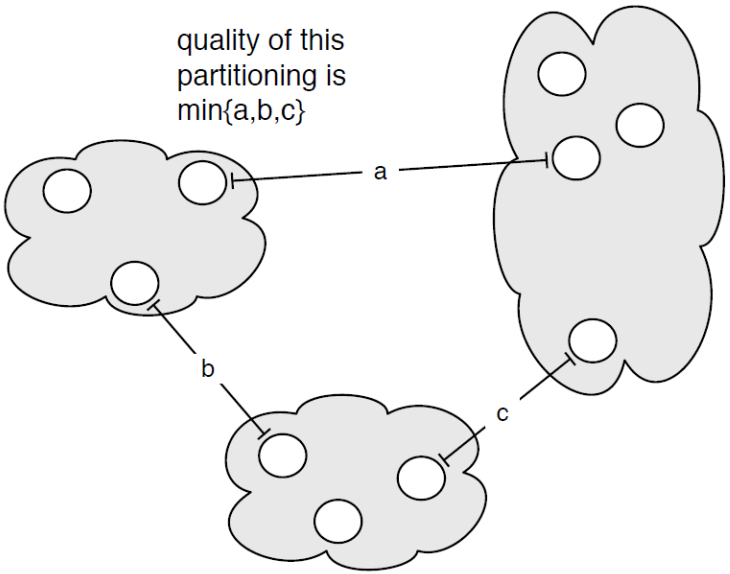
Arora-Mitchell-style "Divide and Conquer", easy to implement in Massively Parallel Computational Models, but bad running time

MST: Single Linkage Clustering

- [Zahn'71] **Clustering** via MST (Single-linkage):

k clusters: remove $\boldsymbol{k-1}$ longest edges from MST

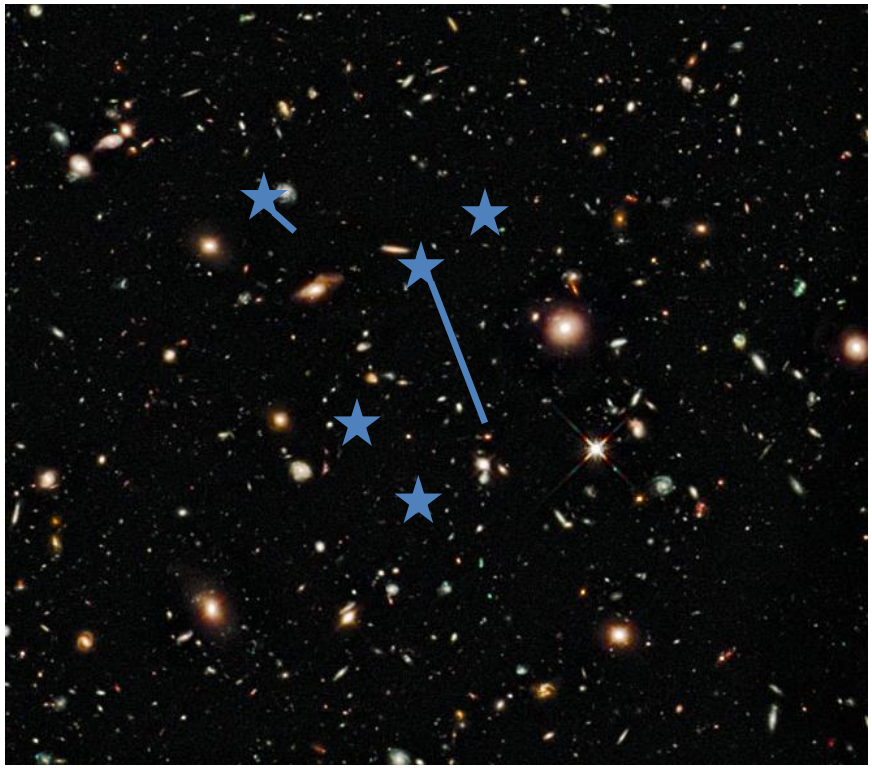
- Maximizes **minimum** intercluster distance



[Kleinberg, Tardos]

Earth-Mover Distance

- Computer vision: compare two pictures of moving objects (stars, MRI scans)



Large geometric graphs

- Graph algorithms: **Dense graphs** vs. sparse graphs
  - **Dense**: $\boldsymbol{S} \gg |V|$.
  - Sparse: $\boldsymbol{S} \ll |V|$.

- **Our setting:**
  - Dense graphs, sparsely represented: $O(\mathbf{n})$ space
  - Output doesn't fit on one machine ($\boldsymbol{S} \ll \boldsymbol{n}$)
- **Today:** $(1+\epsilon)$-approximate MST
  - $\boldsymbol{d} = 2$ (easy to generalize)
  - $\boldsymbol{R} = \log_{\boldsymbol{S}} \boldsymbol{n} = O(1)$ rounds ($\boldsymbol{S} = \boldsymbol{n}^{\boldsymbol{\Omega(1)}}$)

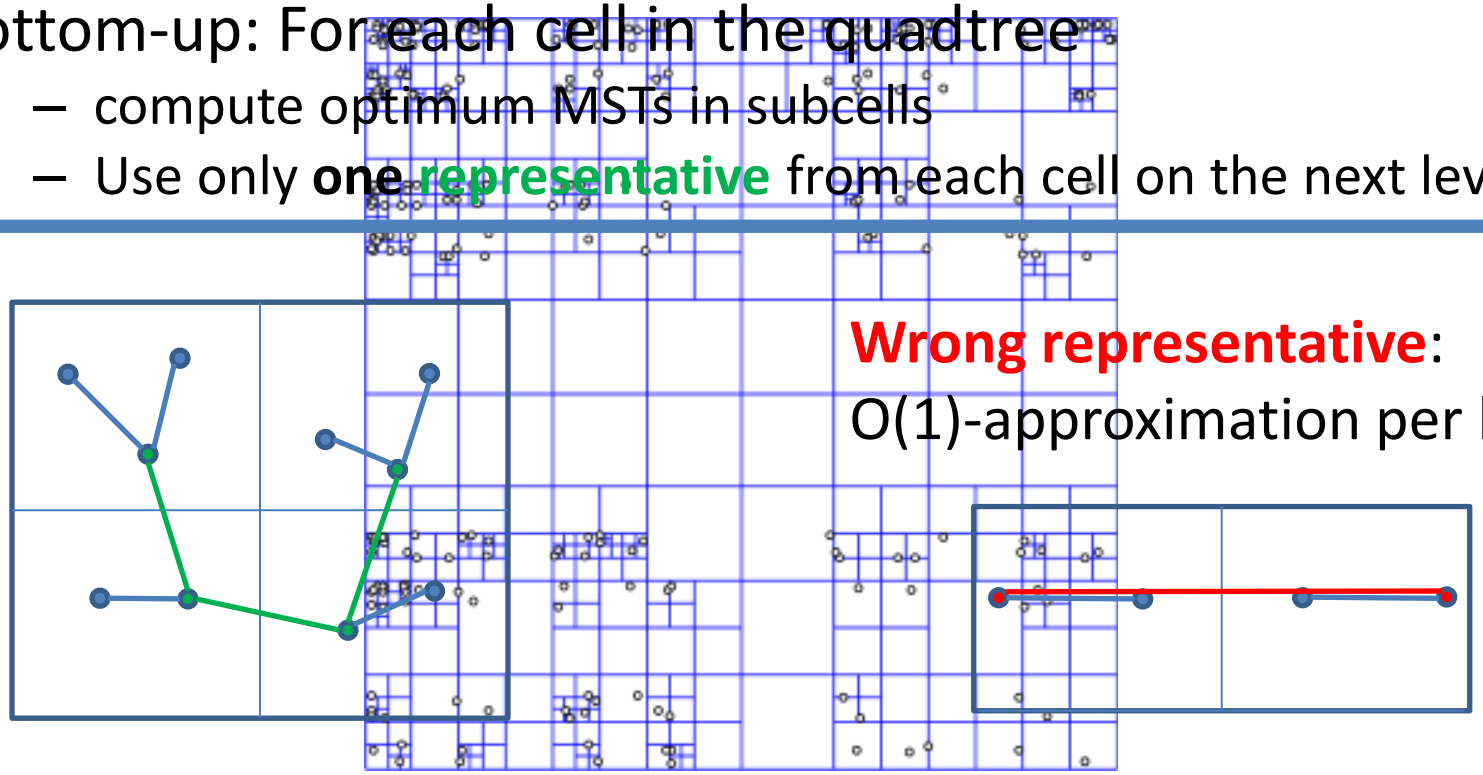
$O(\log n)$-MST in $R = O(\log n)$ rounds

- Assume points have integer coordinates $[0, \dots, \Delta]$, where $\Delta = O(n^2)$.

Impose an $O(\log n)$-depth quadtree

Bottom-up: For each cell in the quadtree

- compute optimum MSTs in subcells
- Use only **one representative** from each cell on the next level



Wrong representative:
O(1)-approximation per level

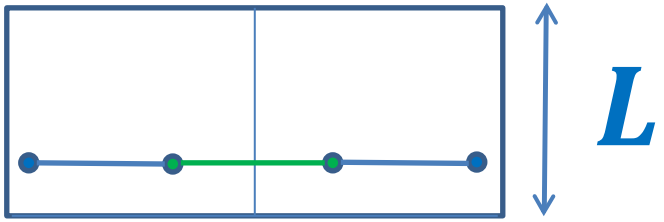
ϵL-nets

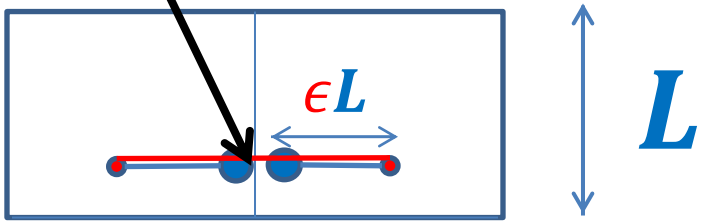
- ϵL-net for a cell C with side length L:
  Collection **S** of vertices in C, every vertex is at distance <= ϵL from some vertex in **S**. (Fact: Can efficiently compute ϵ-net of size $O\left(\frac{1}{\epsilon^2}\right)$)

Bottom-up: For each cell in the quadtree
– Compute optimum MSTs in subcells
– Use ϵL-net from each cell on the next level

- **Idea**: Pay only $O(\epsilon L)$ for an **edge** cut by cell with side L
- Randomly shift the quadtree:
  $\Pr[cut\ edge\ of\ length\ \ell\ by\ L] \leq \ell / L$ – charge errors
  O(1)-approximation per level

Wrong representative





Randomly shifted quadtree

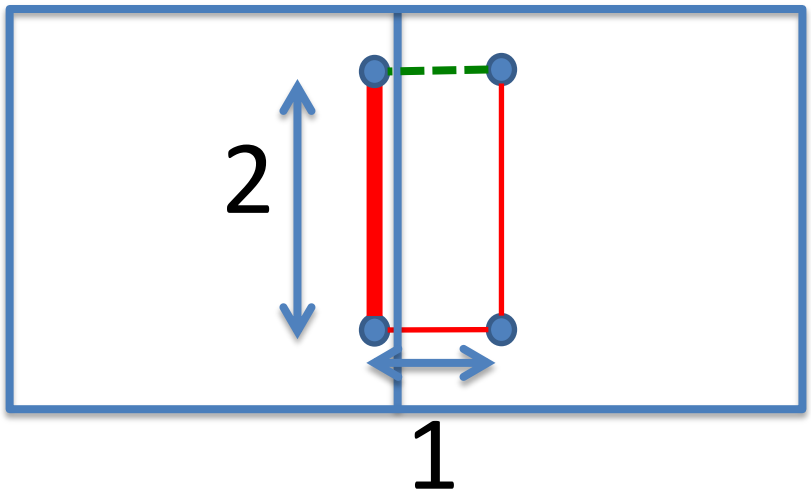
- Top cell shifted by a random vector in $[0, \boldsymbol{L}]^2$

Impose a **randomly shifted** quadtree (top cell length $\boldsymbol{2\Delta}$)

Bottom-up: For each cell in the quadtree

– Compute optimum MSTs in subcells

– Use $\boldsymbol{\epsilon L}$-net from each cell on the next level



Pay **5** instead of **4**

Bad Cut

Pr[Bad Cut] = Ω(1)

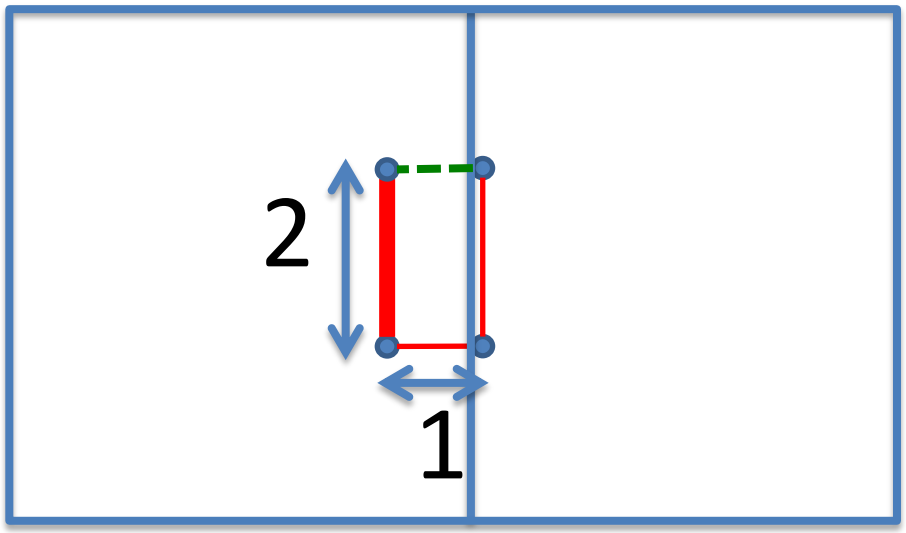
$(1+\epsilon)$-MST in $\mathbf{R} = O(\log\ n)$ rounds

- **Idea:** Only use short edges inside the cells

Impose a **randomly shifted** quadtree (top cell length $\frac{2\Delta}{\epsilon}$)

Bottom-up: For each node (cell) in the quadtree

- compute optimum Minimum Spanning **Forests** in subcells, **using edges of length $\leq \epsilon L$**
- Use only $\epsilon^2 L$-net from each cell on the next level

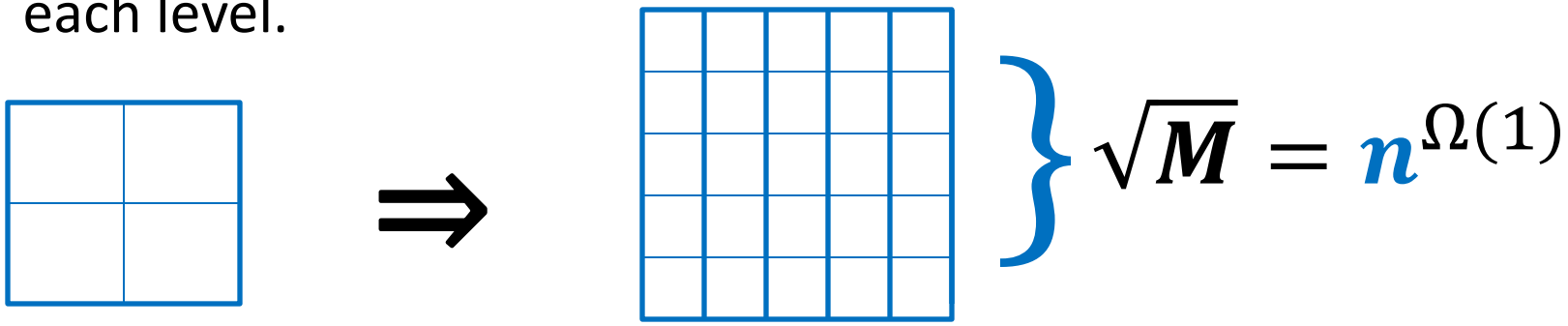


$$L = \Omega\left(\frac{1}{\epsilon}\right)$$

Pr[Bad Cut] = $O(\epsilon)$

$(1+\epsilon)$-MST in $\mathbf{R} = O(1)$ rounds

- $O(\log n)$ rounds => O($\log_S n$) = O(1) rounds
  - Flatten the tree: ($\sqrt{M} \times \sqrt{M}$)-grids instead of (2x2) grids at each level.



Impose a **randomly shifted** ($\sqrt{M} \times \sqrt{M}$)-tree

Bottom-up: For each node (cell) in the tree

- compute optimum MSTs in subcells via edges of length $\leq \epsilon L$
- Use only $\epsilon^2 L$-net from each cell on the next level

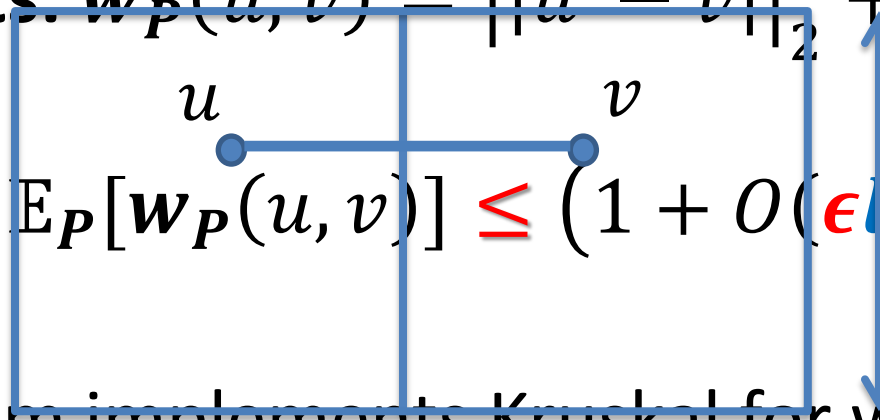
$(1+\epsilon)$-MST in $\mathbf{R} = O(1)$ rounds

Theorem: Let $l = \#$ levels in a random tree P

$$\mathbb{E}_P[\mathbf{ALG}] \leq \big(1 + O(\epsilon l d)\big)\mathbf{OPT}$$

Proof (sketch):

- $\Delta_P(u,v)$ = cell length, which first partitions (u,v)
- **New weights:** $w_P(u,v) = ||u-v||_2 + \epsilon\Delta_P(u,v)$



$$||u-v||_2 \leq \mathbb{E}_P[w_P(u,v)] \leq \big(1 + O(\epsilon l d)\big)||u-v||_2$$

- Our algorithm implements Kruskal for weights w_P

"Solve-And-Sketch" Framework

$(1+\epsilon)$-**MST**:

- "**Load balancing**": partition the tree into parts of the same size
- **Almost linear time locally**: Approximate Nearest Neighbor data structure [Indyk'99]
- Dependence on dimension **d** (size of ϵ-net is $O\left(\frac{d}{\epsilon}\right)^{d}$)
- Generalizes to bounded **doubling dimension**
- Implementation in MapReduce

“Solve-And-Sketch” Framework

$(1+\epsilon)$-**Earth-Mover Distance, Transportation Cost**

- No simple “divide-and-conquer” Arora-Mitchell-style algorithm (unlike for general matching)
- Only recently sequential $(1+\epsilon)$-apprxoimation in $O_\epsilon\left(\boldsymbol{n}\log^{O(1)}\boldsymbol{n}\right)$time [Sharathkumar, Agarwal ‘12]

Our approach (convex sketching):

- Switch to the flow-based version
- In every cell, send the flow to the closest net-point until we can connect the net points

“Solve-And-Sketch” Framework

Convex sketching the cost function for $\boldsymbol{\tau}$ net points

- $F: \mathbb{R}^{\boldsymbol{\tau}-1} \rightarrow \mathbb{R}$ = the cost of routing fixed amounts of flow through the net points
- Function $F' = F\ +$ “normalization” is monotone, convex and Lipschitz, $(1+\boldsymbol{\epsilon})$-approximates F
- We can $(1+\boldsymbol{\epsilon})$-sketch it using a lower convex hull

Thank you! http://grigory.us

Open problems:

- Exetension to high dimensions?
  - Probably no, reduce from connectivity => conditional lower bound： $\Omega(\log n)$ rounds for MST in ℓ_∞^n
  - The difficult setting is $d = \Theta(\log \boldsymbol{n})$ (can do JL)
- Streaming alg for **EMD** and **Transporation Cost**?
- Our work:
  - First near-linear time algorithm for **Transportation Cost**
  - Is it possible to reconstruct the solution itself?

Class Project

- Survey of 3-5 research papers
  - Closely related to the topics of the class
    - Streaming
    - MapReduce
    - Convex Optmization
    - Sublinear Time Algorithms
  - Office hours if you need suggestions
  - Individual or groups of 2 people
  - **Deadline**: December 18, 2015 at 23:59 EST
- Submission by e-mail grigory@grigory.us
  - Submission Email Title: Project + Space + "Your Name"
  - One submission per group listing participants
  - Submission format
    - PDF from LaTeX (best)
    - PDF

Example: Gradient Descent in TensorFlow

- Gradient Descent (covered in class)
- Adagrad: http://www.magicbroom.info/Papers/DuchiHaSi10.pdf
- Momentum (stochastic gradient descent + tweaks): http://www.cs.toronto.edu/~hinton/absps/naturebp.pdf
- Adam (Adaptive + momentum): http://arxiv.org/pdf/1412.6980.pdf
- FTRL: http://jmlr.org/proceedings/papers/v15/mcmahan11b/mcmahan11b.pdf
- RMSProp: http://www.cs.toronto.edu/~tijmen/csc321/slides/lecture_slides_lec6.pdf

K-means Clustering

- Given $\mathrm{X} = \{x_1, \ldots, x_n\} \in \mathbb{R}^d$ find a set of centers $C = (c_1, \ldots, c_k)$ that minimizes

$$\sum_{x \in X} \min_{i \in [k]} \left|\left| x - c_i \right|\right|^2$$

- NP-hard problem
- Popular heuristic local search (Lloyd's alg.)
- For a fixed partitioning $P_1, \ldots, P_k$:

$$c_j = \frac{1}{|P_j|} \cdot \sum_{i \in P_j} x_i$$

Dimension reduction for K-means

- Let $cost_P(X) = \inf_c cost_{P,c}(X)$
- For $0 < \epsilon < \frac{1}{2}$ let $f: X \to \mathbb{R}^n$ be such that

$$\forall i,j: (1-\epsilon)\left|\left|x_i - x_j\right|\right|_2^2 \leq \left|\left|f(x_i) - f(x_j)\right|\right|_2^2 \leq (1+\epsilon)\left|\left|x_i - x_j\right|\right|_2^2$$

- $\hat{P}$ is a γ-approx. clustering for $f(X)$
- P^* is an optimal clustering for X
- **Lemma**.

$$cost_{\hat{P}} \leq \gamma\left(\frac{1+\epsilon}{1-\epsilon}\right) cost_{P^*}(X)$$

Dimension reduction for K-means

- Let $cost_P(X) = \inf_c cost_{P,c}(X)$
- For $0 < \epsilon < \frac{1}{2}$ let $f: X \to \mathbb{R}^{d'}$ be such that

  $\forall i,j: (1-\epsilon)\left|\left|x_i - x_j\right|\right|_2^2 \leq \left|\left|f(x_i) - f(x_j)\right|\right|_2^2 \leq (1+\epsilon)\left|\left|x_i - x_j\right|\right|_2^2$
- $\hat{P}$ is a γ-approx. clustering for $f(X)$
- P^* is an optimal clustering for X
- **Lemma**.

$$cost_{\hat{P}} \leq \gamma\left(\frac{1+\epsilon}{1-\epsilon}\right) cost_{P^*}(X)$$

- $d' = O\left(\log\frac{n}{\epsilon^2}\right)$ suffices by the JL-lemma

Dimension reduction for K-means

- Fix a partition $P = (P_1, \dots, P_k)$

$$cost_P(X) = \sum_{j\in[k]} \sum_{i\in P_j} \left\| x_i - \frac{1}{|P_j|} \sum_{i'\in P_j} x_{i'} \right\|_2^2$$

$$= \sum_{j\in[k]} \frac{1}{|P_j|} \sum_{i\in P_j} \left(\sum_{i'\in P_j} ||x_i||_2^2 - 2\langle x_i, \sum_{i'\in P_j} x_i' \rangle + \left\| \sum_{i'\in P_j} x_i' \right\|_2^2 \right)$$

$$= \sum_{j\in[k]} \frac{1}{|P_j|} \sum_{i\in P_j} \sum_{i'\in P_j} \left(\frac{||x_i||_2^2 + ||x_i'||_2^2}{2} - \langle x_i, x_{i'} \rangle \right)$$

$$\sum_{j\in[k]} \frac{1}{2|P_j|} \sum_{i\in P_j} \sum_{i'\in P_j} \left(||x_i - x_{i'}||_2^2 \right)$$

- $(1-\epsilon) cost_P(X) \le cost_P\big(f(X)\big) \le (1+\epsilon) cost_P(X)$
- $(1-\epsilon) cost_{\hat{P}}(X) \le cost_{\hat{P}}\big(f(X)\big) \le \gamma\, cost_{P^*}\big(f(X)\big) \le \gamma\, cost_{P^*}(X)$

K-means++ Algorithm

- First center uniformly at random from X
- For a set of centers C let:

$$d^2(x, C) = \min_{c \in C} \left|\left|x - c\right|\right|_2^2$$

- Fix current set of centers C
- Subsequent centers: each x_i with prob.

$$\frac{d^2(x_i, C)}{\sum_{x_j \in X} d^2(x_j, C)}$$

- Gives $O(\log k)$-approx. to OPT in expectation

K-means‖ Algorithm

- First center C: sample a point uniformly
- Initial cost $\psi = \sum_x d^2(x, C)$
- For $O(\log \psi)$ times do:
  - Repeat ℓ times (in parallel)
    - C' = sample each $x_i \in X$ indep. with prob.

$$p_x = \frac{d^2(x_i, C)}{\sum_{x_j \in X} d^2(x_j, C)}$$

    - $C \leftarrow C \cup C'$
- For $x \in C$:

  w_x = the #points belonging to this center
- Cluster the weighted points in C into k clusters

K-means‖ Algorithm

- Oversampling factor $\ell = \Theta(k)$
- #points in C: $\ell \log \psi$
- **Thm.** If α-approx. used in the last step then k-means‖ obtains an $O(\alpha)$-approx. to k-means
- If Ψ and Ψ' are the costs of clustering before and after one outer loop iteration then:

$$E[\Psi'] = O(OPT) + \frac{k}{e\ell}\Psi$$

K-means‖ Analysis

- For a set of points $A = \{a_1, \dots, a_t\}$ *centroid* c_A*:*

$$c_A = \frac{1}{|T|} \sum a_t$$

- Order $a_1, \dots, a_T$ in the increasing order by distance from c_A
- Fix a cluster A in OPT
- Fix C prior to the iteration and let:

$$\phi(C) = \sum_x d^2(x, C)$$

$$\phi_A(C) = \sum_a d^2(a, C)$$

- Let $p_t = \frac{d^2(a_t, C)}{\phi(C)}$ be the probability of selecting a_t
- Probability that a_t is the smallest one chosen:

$$q_t = p_t \prod_{j=1}^{t-1} (1 - p_j)$$

K-means‖ Analysis

- Can either assign all points to some selected a_t or keep the original clustering:

$$s_t = \min\left(\phi_A, \sum_{a \in A} ||a - a_t||^2\right)$$

- $E[\phi_A(C \cup C')] \leq \sum_t q_t s_t + q_{T+1}\,\phi_A(C)$

where q_{T+1}= prob. that no point in A is selected

- Simplifying assumption: consider the case when all $p_t = p$ (mean field analysis)
- $q_t = p(1-p)^t$ (decreasing sequence)

K-means‖ Analysis

- $s'_t = \sum_{a \in A} \left|\left|a - a_t\right|\right|^2$
- $\{s'_t\}$ is an increasing sequence

$$\sum_t q_t s_t \leq \sum_t q_t s'_t$$

$$\leq \frac{1}{T}\left(\sum_t q_t \sum_t s'_t\right)$$

$$= \left(\sum_t q_t \cdot \frac{1}{T}\sum_t s'_t\right)$$

$$= \left(\sum_t q_t \cdot 2\,\phi_A^*\right)$$

- $E[\phi_A(C \cup C')] \leq (1 - q_{T+1})\, 2\,\phi_A^* + q_{T+1}\,\phi_A(C)$